NORMAL MODES OF VORTEX VIBRATIONS: VORTICAL HARMONICS REVEALING THEIR NORMAL MODES OF VIBRATION

Abstract. This essay highlights an unusual phenomenon observed in vortices in fluid dynamics: a continuous flow of distorted waves twisting and shrinking while descending toward the center, forming geometric patterns known as logarithmic spirals. In a vortex, these waves represent its normal modes of vibration and can be used as basis functions in series expansions to solve PDEs.

These modes can be identified via the dispersion relation:

$$\omega_{m,k} = \frac{\Gamma}{2\pi a^2}\left(m^2 + (ka)^2\right),$$

describing the natural frequencies of the vortex.

Vortices are solutions of the Euler equations and can be geometrically described by logarithmic spirals in free vortices. All possible spirals correspond to streamlines on the vortex surface, representing wave crests propagating along circular, descending trajectories toward the vortex center.

The functions

$$\eta(\theta, z, t) = A_{m,k} e^{i(m\theta + kz - \omega_{m,k} t)}$$

form a basis series

$$\xi_{m,k}(\theta, z, t) = e^{i(m\theta + kz - \omega_{m,k} t)}, \quad m, k \in \mathbb{Z},$$

used to solve fluid PDEs and develop analytical methods for vortex dynamics.

The orthogonality of the basis functions can be expressed as:

$$\int_0^R \int_0^{2\pi} \xi_{m_i,k_i}(\theta, z)\, \overline{\xi_{m_j,k_j}(\theta, z)}\, d\theta\, dz = 0, \quad \text{for } (m_i, k_i) \neq (m_j, k_j),$$

ensuring proper expansion of the vortex field:

$$\mathbf{u}(\theta, z, t) = A_0 + \sum_{m,k} \left[\xi_{m,k}(\theta, z, t) + \overline{\xi_{m,k}(\theta, z, t)}\right].$$

E sem o teste prévio da sua convergência é a nossa conjectura mais plausível.

Fluid Dynamics Equations

Fluid dynamics can be described using gradient, divergence, and curl of a vector field. The gradient indicates the direction of maximum flow, divergence measures compression or expansion, and curl represents local rotation.

The incompressible Euler equations describe ideal fluid flow:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p,$$

and the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ evolves according to

$$\partial_t \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}.$$

Normal Modes of Vibration

A vortex is a rotational motion of a fluid around an axis or spiral, typically with a central low-pressure region. It can be observed in natural or technological phenomena such as whirlpools and tornadoes.

Logarithmic spiral functions form a series:

$$\eta(\theta, z, t) = \sum_{m,k} \eta_{m,k}(\theta, z, t),$$

providing an analytical framework to solve PDEs in vortex dynamics.

Kelvin Mode Dispersion Equation

$$\omega_{m,k} = \frac{\Gamma}{2\pi a^2}\left(m^2 + (ka)^2\right)$$

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References

(1) G. Guyot, H. Maaloul, A. Archer, *Modeling Vortices with 3D CFD* (2014).
(2) Raphaël Danchin, *Fourier Analysis Methods for the Compressible Navier-Stokes Equations* (2015).
(3) Saffman , *V*ortex Dynamic, Cambridge University Press,1993
(4) Wu et al , *V*orticity and Vortex Dynamic, Spring,2006