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# ASSET PRICING UNDER ANY DISTRIBUTION OF RETURNS: THE OMEGA CAPM

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# Asset Pricing under any Distribution of Returns: The Omega CAPM

Precificação de Ativos sob qualquer Distribuição de Retornos: O Ômega CAPM

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We propose a new version of the Capital Asset Pricing Model (CAPM). This new model works under any distribution of asset returns and its beta is sensitive to variations in the risk-free rate. Furthermore, our model does not make strong assumptions regarding investor behavior.It maintains the simple structure of a single factor and the micro-foundations of the traditional CAPM. This new approach brings several improvements to any models that use the beta as an explanatory variable. After a brief empirical test, we verified that the new model had a better performance than the traditional CAPM. (JEL G12, G11)

**Key words:** CAPM, OCAPM, superior moments.

Este trabalho propõe uma nova versão para o *Capital AssetPricingModel* (CAPM). O novo modelo não faz restrições com relação a distribuições de retorno e seu *beta* é sensível a variações na taxa livre de risco. Além disse não existem pressupostos fortes com relação ao comportamento dos investidores. O modelo mantém a forma simples de um único fator e os micro fundamentos do CAPM original. Este novo enfoque traz melhorias consideráveis para quaisquer modelo que utilizam o *beta* como variável explicativa. Após um breve teste empírico, foi verificado que o modelo aqui proposto possui uma performance superior ao CAPM origina. (JEL G12, G11)

**Palavraschave:** CAPM, OCAPM, momentossuperiores.

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The Mean-Variance model of Markowitz(1952) represented a great innovation inmethods for assets pricing. The central idea of this model is that investorscan make their investment decisions by choosing portfolios that are mean-variance efficient, in the sense that, for each level of expected return (mean) there is no other portfolio of lower risk (variance). According to this approach, combinations of efficient portfolios will create the Efficient Frontier, which represents a financial opportunity set for all risk-averse investors. Thus, on this opportunity setinvestors can choose portfolios according to their preferences for risk. The problem is that, while facing the same opportunity set, investors will choose portfolios based on their utility functions, in such a way that risk will have different prices at each point of the set, given by investors' risk-return substitution rate. The latter being, at this point, equal to the risk-return transformation rate provided by the opportunity set.

Following this approach, the Capital Asset Pricing Model (CAPM), developed by Sharpe (1964), Lintner (1965) and Mossin (1966), has given a reasonable solution to the multiplerisk prices problemby introducing the risk-free asset on the Mean-Variance framework. Considering such asset, the Efficient Frontier was dominated (in terms of risk and return) by the Capital Markets Line (CML), which made it possible to establish a linear relationship between expected returns and a measure of systematic risk, known as'beta', for securities throughout the Securities Market Line (SML). On this extended framework, each asset should return the risk-free rate plus a risk premium given by theproduct of theasset's beta and the uniqueprice of risk.

Although theCAPM is a theoretically rigorous model which can be easily applied, the model requires as a necessary condition a mean-variance efficient market. In order to satisfy this necessary condition, one must impose restrictions on returns distributions or utility functions. The normal distribution and the quadratic utility function were the first known sufficient conditions that could satisfy the mean variance efficiency of the market.

There is a rich literature dealing with the necessary and sufficient conditions of the CAPM. Tobin (1958) believed that all distributions that could be described by two parameters would be enough to achieve the mean variance efficient market, but Feldstein (1969) has shown that the log-normal distribution, which can be described by two parameters, would not be a sufficient condition for the single period CAPM. More conclusive results were given by Ross (1978) with the two funds distributions.He showed that the CAPM could be developed if, for a great number of assets, the return distributions induced the investors to divide their wealth between two portfolios.

The mean variance efficiency of the market may seem like a weak restriction, but it cannot be satisfied if we do not impose some hard restrictions on utility functions or return distributions. Berk (1997) showed that restrictions on distributions would be equivalent to restrict the utility function to a shape that considers only mean and variance. The author also proposes combined restrictions on utility and distribution, e.g. polynomial utility combined with restrictions on superior moments of the distributions (above the second moment).

Therefore, to develop the CAPM one must choose among a great menu of sufficient conditions those that are less restrictive. Empirical distributions of returnsare, in general, skewed and to assume that they are normal or two fund distributions may be unrealistic. However, to impose restrictions on investors behavior is notbetter; there may be other relevant factors to investors other than mean and variance. In fact, for the average investor the mean and the variance may not be relevant at all.

The CAPM also assumes that the beta is sufficient to describe the expected returns. Testing this assumption hasled financial researchersintothe development of many multifactor models. Those models use a combination of empirically tested variables (and sometimes thebeta itself) in order to find the best expression for the expected returns. A large number of variables were found to be statistically significant to explain theexpected returns, such as price-earnings (Basu, 1977; 1983), dividend yields (Litzenberger&Ramaswamy, 1979), company size (Banz, 1981), company book value (Rosemberg*et al.*, 1985) and liquidity (Liu, 2006).Fama and French (1992a) have jointly tested most of these variables and found that the beta had no explanatory power when the expected returns were controlled by size; the authors argue that the size variable may have a higher correlation with the true betas than the estimated betas.This research was the first step to the Famaand French (1993; 1996) five-factor model with two factors for bonds and three factors for stocks. These multifactor models seem to provide a good explanation for the behavior of the expected returns, but they haveneither the theoretical rigor nor the micro-foundations ofthe CAPM.

Some extended version of the CAPM use variables for skewness and kurtosis. Along with mean and variance, they offer a good description of most distributions of return. Kraus and Litzenberger (1976) and Fang and Lai (1997) have respectively tested skewness and kurtosis in theCAPM equation. These extended models have successfully shown that such measures are significant to explain expected returns. However, there are at least two problems with this approach: (i) theextended modelssimply add new variables to theCAPMequation and test them empirically; (ii) it is very unlikely that people actually calculateor even think about 'higher moments' in order to make an investment decision.

Last but not least on the CAPM empirical pitfalls, variance and standard deviation are measures of risk that consider the entire distribution of returns i.e. both good and bad results around mean are equally considered. It is reasonable to assume that risk averse investors are more concerned with the possibility of obtaining returns below average (or below a threshold point) than above average.

In this paper, we propose a new version for the CAPM, named Omega[1] CAPM (OCAPM). This new approach makes no assumptions towards returns distributions and investors behaviorother than greed and risk aversion. Therefore, the discussion regarding necessary and sufficient conditions is much simpler.Furthermore, we use theExpected Shortfall as the risk measure. The Expected Shortfall considers only the downside risk and it is a coherent[2] measure of risk (Arcebi&Tasche, 2002). TheOCAPM is built on the same theoretical rigor as its predecessor, being the beta coefficient the central difference between the two models. TheOCAPM beta is sensitive to variations in the risk-free rate and it considers all the information (not only skewness and kurtosis) of returns distributions. Moreover, the proposed approach does not work with any assumption that, in order to make their investment decisions investors observe or considerskewness and kurtosis.These statistics are actually considered indirectly bythe Omega performance measure, which has a very simple economic interpretation. Finally, theOCAPM maintains the single factor approach ofthe traditional CAPM.

We assume that individuals observe two measures in order to make their investment decisions: (i) the Expected Shortfall i.e. the average value of the loss if individuals actually lose; (ii) the Expected Chance i.e. the average value of the gain, if individuals actually win.These measures are in the core of the OCAPM, as the reward and risk measures used by the model.

In section 1, we present the Omega performance measure of Keating and Shadwick (2002). This measureplays a central role in the OCAPM. Section 2contains the OCAPMtheoretical demonstration. In section 3, we present someempirical tests of the model. Finally, section 4 contains some remarks and ideas for further research.

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[1]The name "Omega" comes from the Omega Performance Measure of Keating and Shadwick (2002).

[2]The Expected Shortfall satisfies the four desirable axioms proposed by Artzner*et al*. (1999) for a risk measure to be coherent. This result is shown by Arcebi and Tasche (2002).

**1.The Omega Performance Measure**

Since our aim is to proposea less restrictive version of theCAPM, regarding to returns distribution and investor utility function, we used the concepts of theOmega performance measure, originally proposed by Keating and Shadwick (2002). This measure makes no assumption towards returns distributions and works considerably well in a two-dimensional space, as in Markowitz framework (1952).

To define the Omega measure, one needs first to assume that a representative investor has an exogenousloss threshold point, $L$, such that for a realization of return, $r$,if $r > L$ the investor is on the gain region and if $r < L$the investor in on the loss region of the returns distribution.

Let $X$ be a random variable (distribution of returns) that assumesvalues $a \leq X \leq b$ and $L$be the loss threshold point. TheOmega function will be defined as:

(1) $$\Omega(L) = \frac{\int_L^b [1 - F_X(x)]dx}{\int_a^L F_X(x)dx}$$

where$F_X(x)$ is the cumulative distribution functionof returns of a portfolio or asset $X$. The Omega measure is the ratio between the gain area and the loss area, as shown in figure 1.

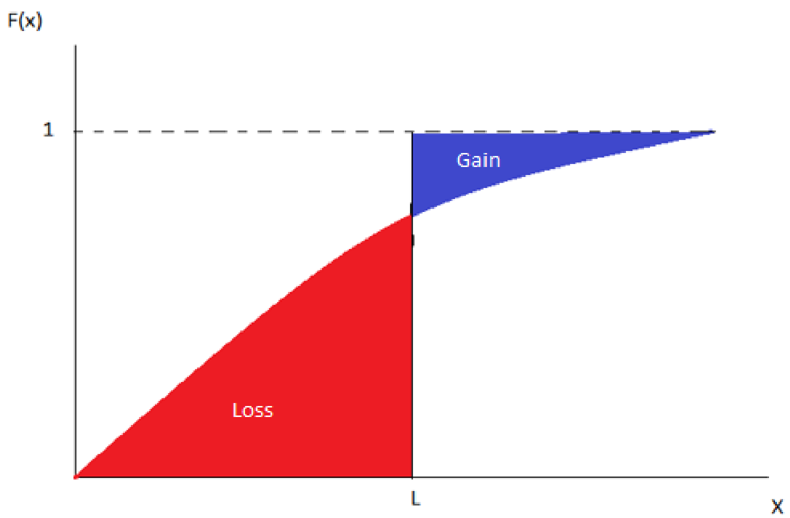


Figure 1: The OmegaMeasure

Kazemi*et al*. (2004) have shown an equivalent representation forthe Omega measure. Defining Expected Chance (EC) as the expected value of gains, conditional to positive results, and Expected Shortfall (ES) as the expected values of loss conditional to negative results, then:

(2) $$\Omega(L) = \frac{\int_L^b (x-L) f_X(x) dx}{\int_a^L (L-x) f_X(x) dx} = \frac{EC(L)}{ES(L)} = \frac{E[Max(X-L;0)]}{E[Max(L-X;0)]}$$

Equation (2) is very similar to the mean-variance model of Makowitz (1952), where Expected Chance representsreward and Expected Shortfall represents risk. Assuming that investors are risk-averse and greedy, they will desire the highest possible EC (it replaces expected return in the mean-variance model) and the lowest ES (it replaces variance in mean-variance model).

The main advantage of the Omega measure is that it considers the entire distribution of return without assuming that investors observe any moments. In fact, the Omega measure is a perfect representation of the distribution, where all information is considered.

The Omega measure does not state that higher moments must be considered directly; instead, it assumes that, in order to make their decisions, investors look at two very simple information: (i) if they win, how much money they will earn on average (EC); (ii) if they lose, how big their loss will be on average (ES). Thus, higher moments ofreturns distributionsare considered indirectly through these measures.

The Omega CAPM, as we will demonstrate, uses theOmega measure to obtain a less restrictive Capital Market Line and Security Market Line.

### **1.1** The Expected Shortfall and the Coherent Measures of Risk

Artzner*et al.* (1999) have defined the four axioms required for a measure of risk to be coherent. Formally, let $R_i$ and $R_j$ be the random returns of assets $i$and $j$, and let $k$ be a constant positive real number, a risk measure $\rho$ will be coherent if it satisfies:

a) Allocation: $\rho(R_i + k) = \rho(R_i) - k, \forall k > 0$.
b) Subaditivity: $\rho(R_i + R_j) \leq \rho(R_i) + \rho(R_j)$.
c) Homogeneity of degree one: $\rho(kR_i) = k\rho(R_i)$.

d) Monotonicity: If for all $R_i$ and $R_j$, $R_i < R_j$ then $\rho(R_i) > \rho(R_j)$

One of the theoretical advantages of the OCAPM regards the measure of risk used by the model. Instead of the variance, we use the Expected Shortfall. The Expected Shortfall satisfies all the four axioms above, as shown by Arcebi and Tasche (2002). The same is not true for the variance and for the standard deviation. The first violates axioms a), c), and d) and the second violates axioms a) and d).

## 2.The Omega Capital Assets Pricing Model

As stated, theOCAPM has the same theoretical rigor as the traditional CAPM as it follows the same steps of Markowitz (1952) and Sharpe (1964) throughout its development. The first step is to show an Omega-based Efficient Frontier. This frontier shows all portfolios that have the smallest Expected Shortfall for a given Expected Chance (or the highest Expected Chance for a given expected Shortfall). Line AB (Figure 2) represents those portfolios for a given loss threshold $L$.

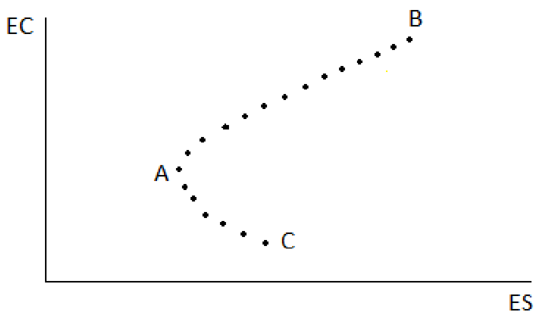

Figure 2: The Omega Efficient Frontier

Assuming that investorsare risk-averse and greedy, theAB set may be obtained by mathematical programming:

(3) $$MaxEC(L) s.t.\ ES(L) = k_1$$

(4) $$MinES(L) s.t.\ EC(L) = k_2$$

where$k_1$ and $k_2$ are constants. The two problems above are equivalent to obtain the set AB. All portfolios on the line between point A and point B are optimal in the Omega approach.

A rational investor will choose a portfolio $p$ in the efficient frontier that makes his (or her) marginal rate of substitution (MRS) equals to the marginal rate of transformation (MRT) at some point of the efficient frontier:

$$MRS_{i\,ES_p}^{\;EC_p} = MRT_{ES_p}^{EC_p}$$

Investors with different utility functions will choose different portfolios depending on their risk aversion, i.e.the Omega Efficient Frontier does not establish a single price for risk. To solve this issue we must include the risk-free asset in the model and proceed to the Omega Capital Markets Line.

### **2.1** TheOmega Capital Markets Line

From this section on, we will treat theloss threshold,$L$, as an opportunity cost of a representative investor. Such interpretation holds because investors will be on the loss region if they obtain returns lower than $L$. Moreover, we assume for now that a representative investor can always obtain a return of $L$ if he chooses to stay out of capital market. TheExpected Chance and the Expected Shortfall of the opportunity $L$ are[3]:

(5) $$EC_L = E[Max(X - L; 0)] = 0$$

(6) $$ES_L = E[Max(L - X; 0)] = 0$$

The expectations above are always zero because the only value$X$ assumes is $L$. To obtain theOmega CML, we have to assume that a representative investor has a portfolio composed by theopportunity cost,$L$, anda portfolio on the efficient frontier, which will be defined as the market portfolio, $m$. Let $\alpha > 0$ be thefraction of wealth invested in the market portfolio and $1 - \alpha$ the fraction of wealth invested in theopportunity $L$ such that:

(7) $$EC_p = E[Max\{\alpha R_m + (1 - \alpha)L - L;\ 0\}]$$

(8) $$ES_p = E[Max\{L - \alpha R_m - (1 - \alpha)L;\ 0\}]$$

[3] The subscripts on the equationsrepresent an asset or portfolio. Although both EC and ES are functions of $L$, $(L)$ will be omitted from now on for means of simplicity.

A function $Max\{.\}$ can be defined as: for $\{X,Y\} \in \mathbb{R}$, $Max\{X,Y\} = \frac{X+Y+|X-Y|}{2}$ (see Trybulec&Bylinski, 1989). Using this definition, equations (7)[4] and (8) can be expressed as:

(9) $$EC_p = E\left[\frac{\alpha R_m - \alpha L + |\alpha R_m - \alpha L|}{2}\right] = E\left[\frac{\alpha(R_m - L) + |\alpha||R_m - L|}{2}\right] = \alpha EC_m$$

(10) $$ES_p = E\left[\frac{\alpha L - \alpha R_m + |\alpha L - \alpha R_m|}{2}\right] = E\left[\frac{\alpha(L - R_m) + |\alpha||L - R_m|}{2}\right] = \alpha ES_m$$

Notice that we can only remove the coefficient $\alpha$ from the expectation operator because it is assumed to be always positive[5].

By differentiating the equations above in terms of $\alpha$ we obtain the slope of the Capital Markets Line in the $EC \times ES$framework, as follows:

(11) $$\frac{\partial EC_p}{\partial \alpha} = EC_m$$

(12) $$\frac{\partial ES_p}{\partial \alpha} = ES_m$$

The slope of the Omega CML will be$\frac{\partial EC_p}{\partial ES_p} = \frac{EC_m}{ES_m}$. The Omega CML equation, on the other handis:

(13) $$EC_p = \frac{EC_m}{ES_m} ES_p$$

Equation (13) shows that the Omega CML starts at the origin ofthe space $EC \times ES$.The Market's Omega measure($\Omega_m = EC_m/ES_m$)is defined as the price of market risk and $ES_p$ is the amount of risk of portfolio $p$. The equilibrium condition is that any portfolio in the Efficient Frontier must have the same Omega measure as the market portfolio.

Notice that themarket portfolio is not any optimal portfolio, but the one where the CML tangencies theEfficient Frontier. The new frontier generated by theOmega CML dominates theOmega Efficient Frontier, as shown in figure 3.

[4] For equation (9) we have $X = \alpha R_m + (1-\alpha)L - L$ and $Y = 0$ and for equation (10) $X = L - \alpha R_m - (1-\alpha)L$and $Y = 0$.

[5]If we differentiate eq. (9) and (10) in terms of $\alpha$, keeping it inside the expectation sign, and evaluate the results for positive values of $\alpha$, we find eq. (11) and (12) either way.

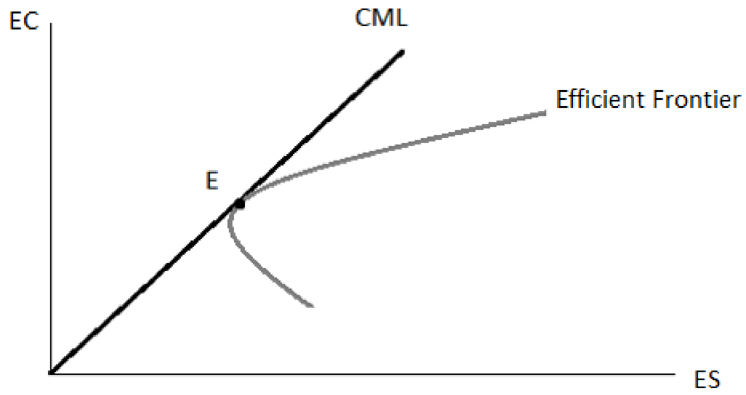


Figure 3: The Omega CML

Point E is theOmega market portfolio. Is has the largest slope among all other portfolios in the Efficient Frontier. Let $W = \{w_1, \dots, w_n\}$ be a set containing fractions of wealth invested on assets $1, \dots, n$. The Market portfolio, $m$, is obtained by the solution of the problem below:

(14) $$Max\Omega_m = Max\frac{EC_m}{ES_m} \; s.t. \; \sum_{i=1}^{n} w_i = 1$$

The two-fund separation theorem is still valid on the Omega approach,and therefore,investors will divide their wealth between the opportunity cost$L$ and the market portfolio.

As in the original CAPM, we have to assume that all investors have homogeneous expectations towards distributions of returns in order to have an equilibrium situation, where the CML is unique. The opportunity cost$L$ must also be unique, the best interpretation being therefore to consider that $L$ is the interest rate at which individuals borrow and lend money. We do not assume, however, that $L$ is constant over time; it might have a stochastic behavior.

## **2.2**The Omega Security Market Line

TheCapital Markets Line allows us to know the equilibrium Expected Chance for a portfolio given its Expected Shortfall, but this is not enough todraw conclusions about individual assets. To evaluate assets, we need to go further until Omega Security Market Line.

Following similar steps from Sharpe (1964), let $m$ be the market portfolio.Supposing that a portfolio $p$has the fraction $\alpha$ of wealth invested in asset $i$ and the fraction $(1-\alpha)$ invested in the market portfolio, the measuresEC and ES of portfolio $p$ are:

(15) $$EC_p = E[Max\{\alpha R_i + (1-\alpha)R_m - L;\ 0\}]$$

(16) $$ES_p = E[Max\{L - (\alpha R_i + (1-\alpha)R_m);\ 0\}]$$

By making the transformation on thefunction $Max\{.\}$ we have:

(17) $$EC_p = E\left[\frac{\alpha R_i + (1-\alpha)R_m - L + |\alpha R_i(1-\alpha) + R_m - L|}{2}\right]$$

(18) $$ES_p = E\left[\frac{L - \alpha R_i - (1-\alpha)R_m + |L - \alpha R_i - (1-\alpha)R_m|}{2}\right]$$

By differentiating both equations above in terms of $\alpha$, we obtain the slope of the Omega Security Market Line. We will show the procedure for equation (17). The development of equation (18) follows the same steps.

$$\frac{\partial EC_p}{\partial \alpha} = \frac{\partial}{\partial \alpha} E\left[\frac{\alpha R_i + (1-\alpha)R_m - L + |\alpha R_i(1-\alpha) + R_m - L|}{2}\right] =$$

$$= E\left[\frac{\partial}{\partial \alpha}\left(\frac{\alpha R_i + (1-\alpha)R_m - L + |\alpha R_i(1-\alpha) + R_m - L|}{2}\right)\right] =$$

(19) $$= E\left[\frac{(R_i - R_m) + \frac{(\alpha R_i + (1-\alpha)R_m - L)}{|\alpha R_i + (1-\alpha)R_m - L|}(R_i - R_m)}{2}\right]$$

Notice that expectation anddifferentiation sign were rearranged using the Leibnitz rule[6]. To advance in the demonstration, we must add and subtract $L$ two times in equation (19):

(20) $$\frac{\partial EC_p}{\partial \alpha} = E\left[\frac{(R_i - L) - (R_m - L) + \frac{(\alpha R_i + (1-\alpha)R_m - L)}{|\alpha R_i + (1-\alpha)R_m - L|}[(R_i - L) - (R_m - L)]}{2}\right]$$

If the market portfolio is optimal on the Omega approach, asset $i$ will already be with optimal proportions in such portfolio so that, if we assume equilibrium, the excess demand for asset $i$ will be zero, i.e. under the market equilibrium $\alpha = 0$. Therefore, by making $\alpha = 0$ in equation (20) we have:

(21) $$\frac{\partial EC_p}{\partial \alpha} = E\left[\frac{(R_i - L) - (R_m - L) + \frac{(R_m - L)}{|R_m - L|}[(R_i - L) - (R_m - L)]}{2}\right]$$

[6] If $f(x,\theta)$, $a(\theta)$ and $b(\theta)$ are differentiable in terms of $\theta$, Cassela and Berger (1990) define the Leibnitz rule as below:

$$\frac{\partial}{\partial \theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx = f(b(\theta),\theta)\frac{\partial}{\partial \theta}b(\theta) - f(a(\theta),\theta)\frac{\partial}{\partial \theta}a(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial \theta} f(x,\theta)\,dx$$

Since in equation (19) the limits of the integral and $\theta$ are independent, the first and the second terms of the Leibnitz rule will be zero.

Since for $|R_m - L| \neq 0$, $\frac{(R_m-L)(R_m-L)}{|R_m-L|} = |R_m - L|$, the final result for the Expected Chance equation is:

(22) $$\frac{\partial EC_P}{\partial \alpha} = \frac{E[R_i]-L-2EC_m+E\left[\frac{(R_m-L)(R_i-L)}{|R_m-L|}\right]}{2} \qquad |R_m - L| \neq 0$$

The results forthe Expected Shortfall are shownbelow in equation (23):

(23) $$\frac{\partial ES_P}{\partial \alpha} = \frac{L-E[R_i]-2ES_m+E\left[\frac{(L-R_m)(L-R_i)}{|L-R_m|}\right]}{2} \qquad |L - R_m| \neq 0$$

The ratio of equations (22) and (23) represents the equilibrium trade-off between EC and ES:

(24) $$\frac{E[R_i]-L-2EC_m+E\left[\frac{(R_m-L)(R_i-L)}{|R_m-L|}\right]}{L-E[R_i]-2ES_m+E\left[\frac{(L-R_m)(L-R_i)}{|L-R_m|}\right]}$$

If we consider that all unsystematic risk is eliminated throughout diversification with no costs, it is reasonable to assume that market will pay a risk premium only for systematic risk. The slope of the Omega CML represents the price of risk for diversified portfolios. Since market pays naught for unsystematic risk, the price of risk represented by expression (24) for $\alpha = 0$ (equilibrium) must be the same price obtained in the Omega CML, as shown by the equation below:

(25) $$\frac{E[R_i]-L-2EC_m+E\left[\frac{(R_m-L)(R_i-L)}{|R_m-L|}\right]}{L-E[R_i]-2ES_m+E\left[\frac{(L-R_m)(L-R_i)}{|L-R_m|}\right]} = \frac{EC_m}{ES_m}$$

Notice that $\frac{(R_m-L)(R_i-L)}{|R_m-L|} = \frac{(L-R_m)(L-R_i)}{|L-R_m|}$ and $\frac{EC_m}{ES_m} = \Omega_m$. Representing $E\left[\frac{(R_m-L)(R_i-L)}{|R_m-L|}\right]$ by $\gamma_i$ and solving equation (25) for $E[R_i]$ we have:

$$E[R_i] - L = \Omega_m(L - E[R_i] + \gamma_i - 2ES_m) - \gamma_i + 2EC_m$$

$$(E[R_i] - L)(1 + \Omega_m) = \Omega_m \gamma_i - 2EC_m - \gamma_i + 2EC_m$$

(26) $$E[R_i] = L + \gamma_i \left[\frac{\Omega_m - 1}{\Omega_m + 1}\right]$$

(27) $$\gamma_i = E\left[\frac{(R_m-L)(R_i-L)}{|R_m-L|}\right]$$

To obtain the final equation of the OCAPM, we must know that $\left[\frac{\Omega_m - 1}{\Omega_m + 1}\right] = \frac{EC_m - ES_m}{EC_m + ES_m}$ and we must show that $EC_m - ES_m = E[R_m] - L$. In order to have a more general result, let $X$ be a random variable such that $a \leq X \leq b$ and $f_X(x)$ represents its density function, then:

(28) $$EC_X = \int_L^b (x - L) f_X(x) dx$$

(29) $$ES_X = \int_a^L (L - x) f_X(x) dx$$

$$EC_X - ES_X = \int_L^b (x - L) f_X(x) dx - \int_a^L (L - x) f_X(x) dx$$

$$= E[X] - L \int_L^b f_X(x) dx - L \int_a^L f_X(x) dx$$

$$= E[X] - L[F_X(b) - F_X(a)]$$

Notice that $[F_X(b) - F_X(a)] = 1$ because the expression considers the entire domain of the distribution of$X$, then $F_X(b) = 1$ and$F_X(a) = 0$. The result of (28) will be:

(30) $$EC_X - ES_X = E[X] - L$$

Our next step is to perform some algebra in the expression $EC_m + ES_m$.Once more, for general results,consider a random variable $X$ such that:

$$EC_X + ES_X = E\left[\frac{X - L + |X - L|}{2}\right] + E\left[\frac{L - X + |L - X|}{2}\right]$$
$$= \frac{E[X] - L + L - E[X] + 2E[|X - L|]}{2}$$

(31) $$EC_X + ES_X = E[|X - L|]$$

Replacing equations (30) and (31) in equation (26) we find the final equation of the Omega CAPM:

$$E[R_i] - L = \gamma_i \left[\frac{\Omega_m - 1}{\Omega_m + 1}\right] = \gamma_i \left[\frac{EC_m - ES_m}{EC_m + ES_m}\right] = \gamma_i \left[\frac{E[R_m] - L}{E[|R_m - L|]}\right] =$$

$$= \frac{\gamma_i}{E[|R_m - L|]} (E[R_m] - L)$$

(32) $$E[R_i] = L + \beta_i (E[R_m] - L)$$

(33) $$\beta_i = \frac{\gamma_i}{E[|R_m - L|]} = E\left[\frac{(R_m - L)(R_i - L)}{|R_m - L|}\right] \frac{1}{E[|R_m - L|]}$$

Equation (33) is theOCAPM beta. Its interpretation is similar to the original CAPM. The constant $L$ is the interest rateat which investors can lend and borrow money; it may or may not be assumed asa risk-free rate[7], but indeed, it represents an opportunity cost for investors. The term $(E[R_m] - L)$ is the price of risk; it is unique as in the original CAPM. The difference between the two models is the $\beta_i$ coefficient, which represents the amount of systematic risk of asset $i$. Although both CAPM beta and OCAPM beta have the same interpretation, the second is not the ratio of covariance and variance, and it considers all returns distribution above and below $L$as well.Furthermore, the OCAPM betas are sensitive to movements in the rate, $L$. Therefore, if$L$, as an opportunity cost for all investors, becomes higher, investors will demand even higher returns from the market and from each asset.

The OCAPM market beta is also equal to one. When themarket presents high returns, ifan asset yields even higher returns, then its beta will be greater than one. Alternatively,if an asset yields lower returns than the market, but those returns still follow the market, its beta will be a number between zero and one. Assets which vary in the opposite direction to the market have negative betas. Thus, theOCAPM beta also represents co-movements between an asset and the market. The main difference between the two models isthat, instead of a mean variance efficient market, we use an Omega efficient market. Therefore, we do not need to assume that investors behave accordingly to utility functions, they need only to be greedy and risk averse. Regarding returns distributions, any distribution can be used as long as the expectations required in the Omega measure exists, and they do exist for any empirical distribution of returns. Another advantage of the OCAPM is that assets can have different distributions, as long as the Omega measure can be defined for them.

Finally, the risk measure we used is theExpected Shortfall, which is a coherent measure of risk in the sense of Artznet*et al* (1999), as shown by Arcebi and Tasche (2002). Therefore, the OCAPM has more realistic assumptions and itmaintains the simple expression and the theoretical rigor of the original CAPM. We do not claim that investors observe and calculate statistical moments directly, in order to consider them in the CAPM equation, like Kraus and Litzemberger (1976) and Fang and Lai (1997). Investors use information such as skewness and kurtosis indirectly when they observe the Expected Chance and the Expected Shortfall.

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[7] The OCAPM risk measure is the Expected Shortfall and equation (6) shows that, on the omega approach, the asset $L$ will always be risk-free. Therefore, when we talk about a risk-free rate, it is intuitive to think about a zero variance rate. When we say that $L$ may or may not be assumed as risk-free, it means that it may or may not have a positive variance.

The graphical representation of the Omega Securities Market Line is shown in figure 4.

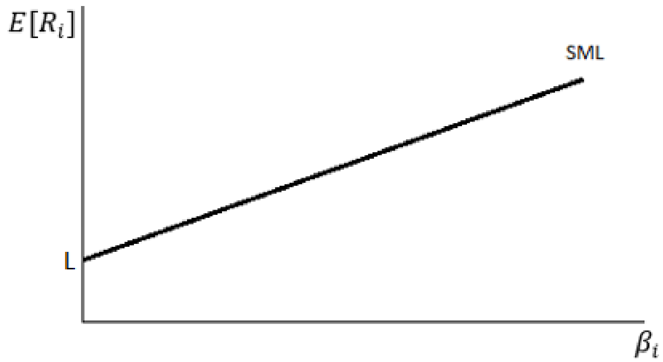


Figure 4: The Omega SML

## 3. Empirical Results

### **3.1** The Data

For this test, we used monthly returns of 395 stock of the Standard &Pours 500 that were available on the entire 2000-2012 period. The data is from the Economatica database. Since we will estimate the market portfolio through optimization, a small database makes the procedure much simpler.

We ranked the 395 stocks by beta (CAPM and OCAPM) and divided them in 15 portfolios to allow as much variation as possible on the betas, and to eliminate some undesirable unsystematic risk. There was no correction for measurement errors in the betas, living the data as raw as possible. Since the measurement errors tend to make the results worst due to problems it creates when we rank the assets, we are free from false good results.

### **3.2** The Market Portfolio

Most of the empirical research regarding the CAPM uses an equally balanced portfolio or an index as a proxy for the market. Since the Roll's critique (1977), it is a consensus that the real market portfolio cannot be estimated and this discussion ceased to be very relevant on the literature. However, the CAPM and the OCAPM requires efficient portfolios on their frameworks. It is impossible to create portfolios with all assets in the economy, but only the feasibleoptimized portfolio, for the assets we choose to use.

We built two market portfolios by optimization, through maximizing the Sharpe ratio[8] and the Omega measure. Table 1 shows some details about the two market portfolios and figure 5 shows the mean-variance efficient frontier with the Omega market portfolio discriminated.

| Market | Mean | Std | Omega | Sharpe Ratio | EC | ES |
|---|---|---|---|---|---|---|
| MeanVariance | 1.987 | 2.869 | 5.489 | 0.692 | 2.427 | 0.442 |
| Omega | 2.174 | 3.372 | 6.271 | 0.589 | 2.583 | 0.412 |

Table 1: Market Portfolios

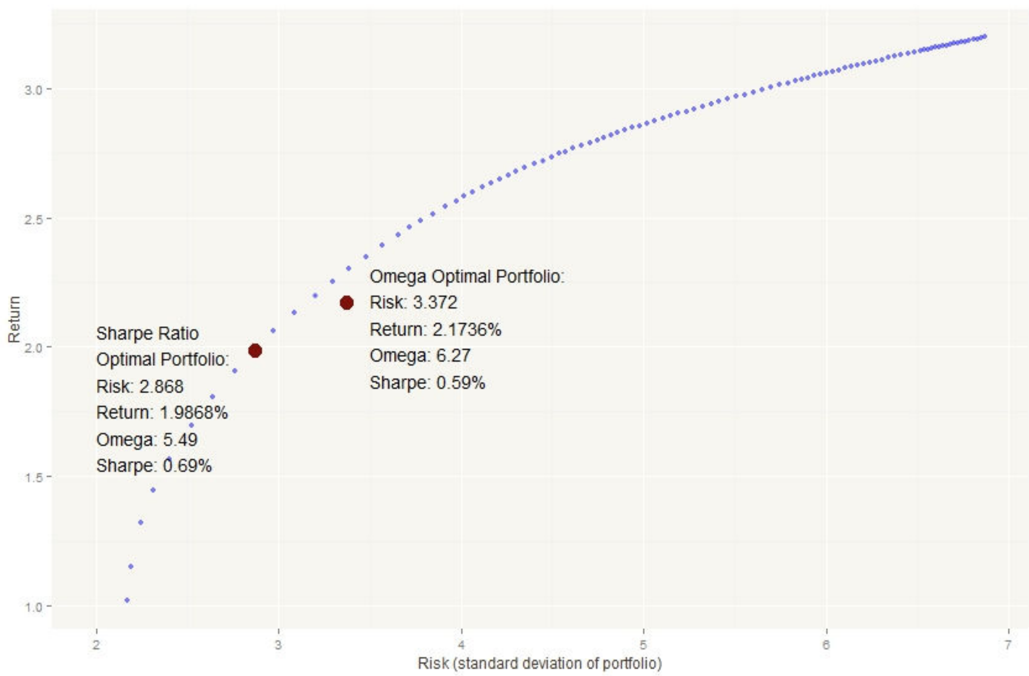


Figure 5: The Efficient Frontier

The Omega portfolio is not in the efficient frontier, its expected return is higher than the mean-variance efficient portfolio, but its standard deviation is also higher. However, the Omega measure is higher in the Omega portfolio. This shows that the optimal portfolio for the Omega measure may be interpreted as bad portfolio in the mean variance framework.

[8]Special Thanks to "The Economist at Large" home page for a very complete and well described R script for the Efficient Frontier. The script is available in http://economistatlarge.com/portfolio-theory/r-optimized-portfolio/r-code-graph-efficient-frontier .

However, the optimal portfolio in the mean variance framework, which has the highest sharp ratio, is suboptimal if we analyze it in the omega framework.

## 2.3 Cross sectional Results

Before discussing empirical results, it is interesting to have some descriptive statistics regarding the ranked portfolios that we created. Table 2 shows the average return and the beta of all portfolios, followed by correlations and some other information.

**Mean Variance Efficient Market**

| portfolio | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Average Returns % | 1.376 | 1.295 | 1.438 | 1.441 | 1.242 | 1.402 | 1.194 | 1.446 | 0.978 | 1.086 | 1.113 | 1.149 | 0.985 | 0.872 | 0.771 |
| CAPM beta | 2.434 | 1.987 | 1.827 | 1.667 | 1.576 | 1.479 | 1.380 | 1.303 | 1.231 | 1.158 | 1.073 | 0.989 | 0.877 | 0.762 | 0.567 |
| OCAPM beta | 1.926 | 1.610 | 1.461 | 1.406 | 1.335 | 1.276 | 1.188 | 1.197 | 1.063 | 1.018 | 0.920 | 0.874 | 0.790 | 0.751 | 0.592 |
| | | | | $E[R_i]$ | $\beta$ | $\Omega\beta$ | | | $E[R_i]$ | $\beta$ | $\Omega\beta$ | | | | |
| Correlations | | | $E[R_i]$ | 1 | 0.787 | 0.850 | | Var. | 0.044 348 | 0.228 098 | 0.118 841 | | | | |
| | | | $\beta$ | 0.787 | 1 | 0.996 | | Std. | 0.210 | 0.477 | 0.344 | | | | |
| | | | $\Omega\beta$ | 0.850 | 0.996 | 1 | | | | | | | | | |

**Omega Efficient Market**

| portfolio | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Average Returns % | 1.994 | 1.540 | 1.490 | 1.519 | 1.302 | 1.313 | 1.156 | 1.339 | 1.076 | 1.008 | 0.866 | 0.988 | 0.842 | 0.643 | 0.673 |
| CAPM beta | 1.798 | 1.561 | 1.364 | 1.330 | 1.218 | 1.152 | 1.084 | 0.989 | 1.082 | 0.912 | 0.792 | 0.743 | 0.640 | 0.593 | 0.503 |
| OCAPM beta | 1.909 | 1.546 | 1.393 | 1.325 | 1.243 | 1.182 | 1.119 | 1.068 | 1.008 | 0.936 | 0.875 | 0.811 | 0.730 | 0.646 | 0.496 |
| | | | | $E[R_i]$ | $\beta$ | ob | | | $E[R_i]$ | $\beta$ | $\Omega\beta$ | | | | |
| Correlations | | | $E[R_i]$ | 1 | 0.964 | 0.960 | | var | 0.126 | 0.126 | 0.126 | | | | |
| | | | $\beta$ | 0.964 | 1 | 0.991 | | std | 0.355 | 0.355 | 0.355 | | | | |
| | | | $\Omega\beta$ | 0.960 | 0.991 | 1 | | | | | | | | | |

Table 2: Some information of the ranked portfolios

The most important information of table 2 is the correlation matrix. Both the CAPM and the OCAPM beta are highly correlated, and their correlation with the expected return vector is also high. The correlation between the betas is 0.996 for the mean variance efficient market and 0.991 for the omega efficient market.

We estimated the CAPM and the OCAPM on both markets (mean variance efficient market and omega efficient market) for a better comparison between the two models. The goal here is to check if the linear relationship between the expected return and the betas is stronger in any of the models. Tests for the sufficiency of the beta is left for another paper. Table 3 summarizes the results. The regressions are simple cross section of the expected return and the betas; we also test the CAPM and the OCAPM betas together in the regression.

| | | MeanVariance Market | | | Omega Market | | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 1 | 2 | 3 |
| CAPM | coef.% | 0.347 | | -1.167 | 0.964 | | -0.160 |
| | t | (4.26) | | (-1.23) | (14.49) | | (-0.38) |
| | p-value | (0.001) | | (0.241) | (0.000) | | (0.710) |
| OCAPM | coef.% | | 0.493 | 2.105 | | 0.977 | 1.135 |
| | t | | (4.39) | (1.64) | | (20.09) | (2.87) |
| | p-value | | (0.001) | (0.128) | | (0.000) | (0.014) |
| Cons | coef.% | 0.716 | 0.614 | 0.323 | 0.170 | 0.123 | 0.120 |
| | t | (6.55) | (4.88) | (1.31) | (2.36) | (2.00) | (1.97) |
| | p-value | (0.000) | (0.000) | (0.216) | (0.034) | (0.066) | (0.072) |
| $R^2$ | | 0.62 | 0.65 | 0.7 | 0.93 | 0.95 | 0.95 |
| n | | 15 | 15 | 15 | 15 | 15 | 15 |

-Regressions with number 1 are for the CAPM, number 2 for the ocapm and number 3 for both betas together.

Table 3: Cross Sectional Regressions

Surprisingly, the OCAPM performed better even on the mean variance efficient market, with higher t-statistic, coefficient and $R^2$. The mean variance efficient market regressions showed a linear relationship between betas and expected returns much flatter than expected (the excess returns of the market calculated from the database was 1.98%). The third regression showed a negative signal for the CAPM beta, and a strong multicolinearity between the two betas.

The Omega efficient market regressions also had a flatter relationship between expected returns and beta (excess return of the omega efficient market of 2.17%), but this relationship was stronger than the one obtained on the mean variance efficient market. The negative signal for the CAPM beta persisted on the extended regression, and even though both betas are highly correlated, the OCAPM beta remained significant. The $R^2$ of 0.95 on regressions 2 and 3 indicate that the CAPM beta adds no information regarding the expected returns when those are controlled by the OCAPM beta. Another interesting observation regards the constant term in all regressions of the Omega market: it was not significant, since the dependent variable is excess return; so, a null constant is a better result.

## IV. Final Remarks

In this paper, we proposed a new version for the CAPM named Omega CAPM (OCAPM). In the considered model, we make no assumptions regarding distributions of returns, and the only assumptions regarding utility functions are greed and risk aversion.

We have followed every step of the original CAPM demonstration in order to maintain its micro-foundations and its theoretical rigor. The only structural difference between the two models is in the way we calculate their betas.

The OCAPM approach considers information of higher moments, such as skewness and kurtosis, but we do not claim that individuals observe these statistics in order to make their investment decisions. Instead, they observe two measures, which have very simple economic interpretation i.e. the Expected Chance and the Expected Shortfall. Respectively, they represent how much money the investor will earn, on average, given that he (or she) won, and how much he (or she) will lose, on average, given that he (or she) lost. The Expected Shortfall is also the risk measure used in the OCAPM. It is a coherent measure of risk (Arcebi&Tasche, 2002) and it considers as risk only the downside risk (risk of losing).

The OCAPM maintains the single factor simplicity of its predecessor. The sufficiency of the beta coefficient is also assumed, but we acknowledge it deserves further research. We believe that even if the beta sufficiency does not verify, it may bring a significant improvement to models that use the beta as an explanatory variable.

Empirically, we performed a brief test comparing the CAPM and the OCAPM using optimized market portfolios. The objective was not to reject or accept any of the models, but to verify which of them presented a stronger linear relationship between beta and expected returns. The results showed that the OCAPM performed better on both optimized market portfolios (mean variance and omega). The results were better for both models (the CAPM and the OCAPM) on the Omega efficient market.

## References


ACERBI, Carlo; TASCHE, Dirk. On the coherence of expected shortfall. **Journal of Banking & Finance**, v. 26, n. 7, p. 1487-1503, 2002.

ARTZNER, Philippe; DELBAEN, Freddy; EBER, Jean-Mark; HEATH, David. Coherent measures of risk. **Mathematical finance**, v. 9, n. 3, p. 203-228, 1999.

BANZ, Rolf W. The relationship between return and market value of common stocks. **Journal of financial economics**, v. 9, n. 1, p. 3-18, 1981.

BASU, Sanjoy. Investment performance of common stocks in relation to their price-earnings ratios: A test of the efficient market hypothesis. **The Journal of Finance**, v. 32, n. 3, p. 663-682, 1977.

BASU, Sanjoy. The relationship between earnings' yield, market value and return for NYSE common stocks: Further evidence. **Journal of financial economics**, v. 12, n. 1, p. 129-156, 1983.

BERK, Jonathan B. Necessary conditions for the CAPM. **Journal of Economic Theory**, v. 73, n. 1, p. 245-257, 1997.

BLACK, Fischer. Capital market equilibrium with restricted borrowing. **The Journal of Business**, v. 45, n. 3, p. 444-455, 1972.

CASELLA, George; BERGER, Roger L. **Statistical inference**. Belmont, CA: Duxbury Press, 1990.

CHEN, NAI-FU. Some empirical tests of the theory of arbitrage pricing.**The Journal of Finance**, 38.5 (1983): 1393-1414.

FAMA, Eugene F.; FRENCH, Kenneth R. Common risk factors in the returns on stocks and bonds. **Journal of financial economics**, v. 33, n. 1, p. 3-56, 1993.

FAMA, Eugene F.; FRENCH, Kenneth R. Multifactor explanations of asset pricing anomalies. **The journal of finance**, v. 51, n. 1, p. 55-84, 1996.

FAMA, Eugene F.; MACBETH, James D. Risk, return, and equilibrium: Empirical tests. **The Journal of Political Economy**, p. 607-636, 1973.

FANG, Hsing; LAI, Tsong-Yue. Co-kurtosis and Capital Asset Pricing.**Financial Review**, v. 32, n. 2, p. 293-307, 1997.

FELDSTEIN, Martin S. Mean- variance analysis in the theory of liquidity preference and portfolio selection. **The Review of Economic Studies**, v. 36, p. 5-12, 1969.

KAZEMI, Hossein; SCHNEEWEIS, Thomas; GUPTA, Bhaswar. Omega as a performance measure. **JOURNAL OF PERFORMANCE MEASUREMENT.**, v. 8, p. 16-25, 2004.

KEATING, Con; SHADWICK, William F. A universal performance measure.**Journal of performance measurement**, v. 6, n. 3, p. 59-84, 2002.

KRAUS, Alan; LITZENBERGER, Robert H. SKEWNESS PREFERENCE AND THE VALUATION OF RISK ASSETS*. **The Journal of Finance**, v. 31, n. 4, p. 1085-1100, 1976.

LINTNER, John. The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. **The review of economics and statistics**, v. 47, n. 1, p. 13-37, 1965.

LITZENBERGER, Robert H.; RAMASWAMY, Krishna. The effect of personal taxes and dividends on capital asset prices: Theory and empirical evidence.**Journal of financial economics**, v. 7, n. 2, p. 163-195, 1979.

LIU, Weimin. A liquidity-augmented capital asset pricing model. **Journal of financial Economics**, v. 82, n. 3, p. 631-671, 2006.

MARKOWITZ, Harry. Portfolio selection*. **The journal of finance**, v. 7, n. 1, p. 77-91, 1952.

MERTON, Robert C. An intertemporal capital asset pricing model.**Econometrica: Journal of the Econometric Society**, p. 867-887, 1973.

MOSSIN, Jan. Equilibrium in a capital asset market. **Econometrica: Journal of the Econometric Society**, p. 768-783, 1966.

REINGANUM, Marc R. The arbitrage pricing theory: some empirical results.**The Journal of Finance**, v. 36, n. 2, p. 313-321, 1981a.

ROLL, Richard. A critique of the asset pricing theory's tests Part I: On past and potential testability of the theory. **Journal of financial economics**, v. 4, n. 2, p. 129-176, 1977.

ROLL, Richard; ROSS, Stephen A. An empirical investigation of the arbitrage pricing theory. **The Journal of Finance**, v. 35, n. 5, p. 1073-1103, 1980.

ROSENBERG, Barr; REID, Kenneth; LANSTEIN, Ronald. Persuasive evidence of market inefficiency. **The Journal of Portfolio Management**, v. 11, n. 3, p. 9-16, 1985.

ROSS, Stephen A. Mutual fund separation in financial theory—the separating distributions. **Journal of Economic Theory**, v. 17, n. 2, p. 254-286, 1978.

ROSS, Stephen A. The arbitrage theory of capital asset pricing. **Journal of economic theory**, v. 13, n. 3, p. 341-360, 1976.

SHARPE, William F. Capital asset prices: a theory of market equilibrium under conditions of risk*. **The journal of finance**, v. 19, n. 3, p. 425-442, 1964.

TOBIN, James. Liquidity preference as behavior towards risk. **The Review of Economic Studies**, v. 25, n. 2, p. 65-86, 1958.

TRYBULEC, Andrzej; BYLINSKI, Czesław. Some properties of real numbers operations: min, max, square, and square root. **Journal of Formalized Mathematics**, v. 1, n. 198, p. 9, 1989.