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Existence Results for nonlinear elliptic problems with free nonlinearities

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"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."(Neumann, John v.)

RESUMO

O objetivo deste trabalho é estudar, dentro do campo de equações diferenciais parciais, problemas elípticos onde podemos identificar algum tipo de criticalidade no comportamento da função não linear presente e, ao final de cada um dos três problemas principais apresentados aqui, buscar a existência de soluções estritamente positivas para os mesmos.

No primeiro capítulo, apresentaremos a leitor uma breve história dos problemas que buscamos estudar e as noções de crescimento crítico de Sobolev e de Trudinger-Moser, noções que se diferenciam principalmente pelo operador elíptico considerado, pelos espaços de funções em que procuramos soluções e, adicionalmente, pelos métodos que empregamos. São estas características que moldam as principais complicações que tivemos de enfrentar para a resolução dos problemas postos.

No segundo capítulo, olhamos para o primeiro problema de nosso interesse, a saber o problema de condição de fronteira mista,

$$\begin{cases} -\Delta u = \lambda u^{q-1} + f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ B(u) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

onde $B(u)$ é um operador de fronteira mista de Dirichlet-Newmann, combinando duas diferentes noções de condição de fronteira. Neste caso, a criticalidade da função f é dada pelo expoente crítico de Sobolev, $2^* = \frac{2N}{N-2}$, onde N é a dimensão do espaço em que Ω se encontra.

Em seguida, no terceiro capítulo, olhamos para um sistema elíptico acoplado,

$$\begin{cases} -\Delta u - \phi u^{2^*-2} = \frac{\lambda}{u^\gamma} & \text{in} \quad \Omega, \\ -\Delta \phi = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2}$$

e o fato de ainda considerarmos o operador Laplaciano implica novamente em uma condição de crescimento crítico de Sobolev, de modo que tomamos f abaixo de uma múltipla da curva dada por u^{2^*}. Vemos que este crescimento também está presente na primeira equação, além da consideração de uma singularidade como parte da não-linearidade.

Por fim, no quarto e último capítulo, consideramos enfim um problema com o operador elíptico não linear, N-Laplaciano,

$$\begin{cases} -\Delta_N u - \phi \frac{f(u)}{u} = \frac{\lambda}{u^\gamma} & \text{in} \quad \Omega, \\ -\Delta_N \phi = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3}$$

Novamente tratamos um sistema, sendo este bem similar ao primeiro. O operador, porém, nos força a considerar a condição de criticalidade de Trudinger-Moser, sendo que agora incorporamos também a função f à primeira equação.

Mais detalhes sobre os problemas tratados, os operadores e suas noções de criticalidade serão fornecidos no devido tempo, assim como os métodos de resolução dos mesmos. Utilizaremos aqui os métodos não-variacionais de Galerkin e da Teoria de Ponto Fixo de Schauder.

Palavras-chave: Equações Elípticas. Crescimento Crítico. Expoente crítico de Sobolev. Desigualdade de Trudinger-Moser. Sistema Schrodinger-Poisson. Método de Galerkin. Teoria do Ponto Fixo de Schauder.

ABSTRACT

The main objective of the present work is to study, within the field of partial differential equations, elliptic problems where we can identify some form of criticality in the behavior of the nonlinear function present and, at the end of each of the three appointed problems, to prove the existence of strictly positive solutions to such.

In the first chapter, we present to the reader a brief historical vision of the problems we seek to study and the notions of critical growth in the sense of Sobolev and in the sense of Trudinger-Moser, which differ from one another mainly by the considered elliptic operator, by the function spaces in which we look for solutions and, additionally, by the methods we employ. This are the factors that summon the main complications we have encountered while resolving the proposed problems.

In the second chapter, we look at our first problem considered, namely the mixed boundary condition problem,

$$\begin{cases} -\Delta u = \lambda u^{q-1} + f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ B(u) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

where $B(u)$ is a Dirichlet-Neumann mixed boundary operator, which combines the two different notions of boundary condition. In this case, the critical behavior of the function f is given by the Sobolev critical exponent, $2^* = \frac{2N}{N-2}$, where N is the dimension of the space where Ω resides.

Following that, in our tird chapter, we look at an elliptic system highly coupled,

$$\begin{cases} -\Delta u - \phi u^{2^*-2} = \frac{\lambda}{u^\gamma} & \text{in} \quad \Omega, \\ -\Delta \phi = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{5}$$

and the fact that we still treat the Laplacian operator implies once more that the critical growth condition is given by the Sobolev critical exponent, so that we take f below (but still able to achieve the growth of) the curve u^{2^*}. One may notice that this growth condition is also seen in the first equation, joined with the presence of a singular term as part of the nonlinearity.

At last, in the fourth and final chapter, it is considered a problem with the nonlinear elliptic operator, the N-Laplacian,

$$\begin{cases} -\Delta_N u - \phi \frac{f(u)}{u} = \frac{\lambda}{u^\gamma} & \text{in} \quad \Omega, \\ -\Delta_N \phi = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{6}$$

We see that once more we treat a system, quite similar even to the first. The operator, however, forces us to consider the condition of criticality of Trudinger-Moser, whereas we also incorporate the same function f to the first equation.

Additional details about the treated problems, their operators and the two notions of critical growth will be given in time, as will be done for the methods used to solve them. We will make use, here, of the Galerkin Method and the Schayder Fixed Point Theorem, both comprising a non variational approach to the resolution of elliptic problems. Furthermore, the important results of each chapter

Keywords: Elliptic Problems. Critical Growth. Sobolev Critical Exponent. Trudinger-Moser Inequality. Schrodinger-Poisson System. Galerkin Method. Schauder Fixed Point Theorem.

SUMMARY

1 INTRODUCTION

1.1 MAIN GOALS

In this work, we seek to obtain existence and positiveness results for three different classes of elliptic problems, each one of them containing certain elements which hinder the use of the more common methods, such as variational ones. Our main goal is to show how we can expand on the results of the current literature with regard to elliptic problems by considering free nonlinearities obeying critical or supercritical growth, both in the sense of Sobolev and of Trudinger-Moser, which shall be defined later. By free nonlinearity, we mean restricting our functions only in their growth, keeping their behaviour quite general. In this context, we have then chosen to apply the non variational Galerkin Method, also to be introduced ahead, to the resolution of such elliptic problems. The choice of our main problems were due to the difficulties the literature encountered when treating them, aside from their external motivation, more apparent in the last two cases, which we shall pass through briefly.

1.2 PRELIMINARY SPACES AND DEFINITIONS

Let us pass briefly through some definitions which shall be of great importance throughout this entire work.

- Ω will, unless explicitly stated otherwise, denote a subset of $\mathbb{R}^N$, which will have its dimension specified when necessary, being a smooth bounded and open set, that is, a smooth domain.
- $C^k(A)$, for $k = 1, 2, 3, \cdots$ and a subset $A \subset \mathbb{R}^N$, is the space of functions $u : A \to \mathbb{R}$ for which its derivatives up to order k exist and are continuous.
- $C^\infty(A)$ is the space of functions $u : A \to \mathbb{R}$ for which its derivatives up to any order exist and are continuous. Functions in $C^\infty(A)$ are also called *smooth functions*.
- $C_0^\infty(A)$ is the subspace of $C^\infty(A)$ given by smooth functions which vanish outside a compact set contained in A. The closure of the set of points in $\mathbb{R}^N$ for which $u \in C_0^\infty(A)$ does not vanish is called the support of u, denoted by *supp* u.
- $L^p(\Omega)$, for $p \in [1, +\infty)$, denotes the space of measurable functions for which the p-th power of its module is integrable,
  $$L^p(\Omega) := \left\{ u\colon \Omega \to \mathbb{R}\,; u \text{ is measurable and } |u|_p = \left(\int_\Omega |u|^p \; dx \right)^{1/p} < +\infty \right\}$$
  We have purposely written the integral above as $|u|_p$ because it defines a norm in $L^p(\Omega)$. For the case when $p = +\infty$, we define $L^p(\Omega) = L^\infty(\Omega)$ as the space of

measurable functions for which the quantity

$$\operatorname*{ess\,sup}_{x\in\Omega}|u(x)| = \inf\{C>0\ ;\ |u(x)|\leq C \ \text{ a. e. in } \Omega\}$$

is finite. In other words, it is the space of measurable functions which are bounded almost everywhere.

- Given a function $u \in L^p(\Omega)$, if there exists a function $g \in L^{p'}(\Omega)$ such that, for all $\varphi \in C_0^\infty(\Omega)$, we have
$$\int_\Omega u\frac{\partial\varphi}{\partial x_i}\,dx = \int_\Omega g\varphi\,dx,$$
then we say that g is the weak derivative of u in relation to x_i. We write $g = \frac{\partial u}{\partial x_i}$, since if g is the derivative of u in the classical sense, it will also be a weak derivative. In the same way, we can generalize this definition for higher order weak derivatives denoting, for some multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, $D^\alpha u$ its weak derivative of order α.[1]

- $W^{k,p}(\Omega)$, for $p \in [1, +\infty)$ and $k \in \mathbb{N}$, is the space of p-integrable functions such that its weak derivatives of k-th power are also p-integrable.
$$W^{k,p}(\Omega) := \{u \in L^p(\Omega)\ ; D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leq k\}.$$
We endow it with the norm
$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k} |D^\alpha u|_p^p\right)^{1/p}.$$

- $W_0^{k,p}(\Omega)$, for $p \in [1, +\infty)$ and $k \in \mathbb{N}$ is the closure of $C_0^\infty(\Omega)$ relative to the $W^{k,p}(\Omega)$ norm.

- $H^k(\Omega)$, for $k \in \mathbb{N}$, will be the label we give to the crucial space $W^{k,2}(\Omega)$. Furthermore, we write $H_0^k(\Omega)$ for $W_0^{k,2}(\Omega)$, so that, for the most often needed Sobolev space, $W_0^{1,2}(\Omega)$, we use $H_0^1(\Omega)$.

1.3 CRITICAL EXPONENTS IN ELLIPTIC EQUATIONS

In this first section, we will study the aspects of an elliptical problem which makes it critical in its conditions, so that we can, in the following chapters, study some important cases. Firstly, we shall work with the linear operator $-\Delta$, for which the critical growth is determined by the limit in the exponent of the Sobolev Embedding Theorems (consult the

[1] More specifically, D^α should be understood to be a partial derivative in the *distributional sense*, requiring thus the study of distribution theory. We shall not deepen in this study, since this higher formalism is not required, but one can check [1] for more details.

Appendix for more details). After that, we will analyze the boundary problem with the p-Laplacian operator, in which case the criticality is given by the so called Trudinger-Moser inequality. Both this concepts will be made clear in a moment.

Let us consider first the following elliptic problem

$$\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Problem (1.1), characterized by the second order linear operator

$$-\Delta u = div(\nabla u) = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$$

and coupled with the Dirichlet boundary condition, requires the use of the Sobolev Space $H_0^1(\Omega)$.

Remark 1.3.1. The most common and accepted approach to problems of this sort is to divide it in two step: obtaining what is called a weak solution and proving its regularity. What we mean by a weak solution (the interested and unfamiliar reader can be referred to [2, 3] for a deeper look in the matter) is a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v \, dx = \int_\Omega f(u) v \, dx \ , \quad \forall \ v \in H_0^1(\Omega).$$

Because of our constant use of the term, we shall refer to such a function u as a solution to Problem (1.1), calling it a *classical solution* if we are able to prove $u \in C^2(\overline{\Omega})$.

We will now analyze the conditions on f that guarantee the existence of solution to Problem (1.1). Under more strict hypothesis, those which we shall wish to generalize in the following chapters, this problem can be treated by variational methods. Firstly, suppose we ask that f satisfies

(h_1) The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and bounded.

Given this, the functional associated with Problem (1.1) is

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \ dx - \int_\Omega F(u) \ dx = \frac{1}{2} \|u\|^2 - \int_\Omega F(u) \ dx \ , \quad u \in H_0^1(\Omega), \tag{1.2}$$

where $F(s) = \int_0^s f(s)ds$ is the primitive of f, which we know to be a continuous function.

As it can be easily seen, I defined in this way is found to be coercive and bounded below. This can be proven by use of the best Sobolev constant, which permits us estimate the norm of a function u in $L^q(\Omega)$ by its norm in $H_0^1(\Omega)$. With it, and noting that, by (h_1), we have

$$|F(s)| \leq a\,|s|\,, \quad \forall s \in \mathbb{R},$$

we can write, for every $u \in H_0^1(\Omega)$,

$$I(u) \geq \frac{1}{2}\|u\|^2 - C\|u\|. \tag{1.3}$$

With this, we prove the existence of the value $m = \inf\{I(u)\ ; u \in H_0^1(\Omega)\}$ and, therefore, a minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$. We only need to prove then the existence of a limit $u \in H_0^1(\Omega)$ for u_n.

Theorem 1.3.1. *If f is a function satisfying (h_1), then there exists a solution to Problem* (1.1).

Proof. If $\{u_n\}_{n\in\mathbb{N}}$ is a minimizing sequence for I, we readily see that it must be bounded in $H_0^1(\Omega)$, since $I(u_n)$ is bounded. By the Sobolev Embedding Theorems, there exists a function $u_* \in H_0^1(\Omega)$ and a subsequence of (u_n) (we shall, with an abuse of notation, still denote this subsequence by u_n) such that

$$\begin{cases} u_n \rightharpoonup u_* \text{ in } H_0^1(\Omega), \\ u_n \to u_* \text{ in } L^q(\Omega), \text{ for } q \in [1, 2^*) \\ u_n \to u_* \text{ a.e. in } \Omega. \end{cases} \tag{1.4}$$

Now, by the continuity of F, the third conclusion in (1.4) implies $F(u_n) \to F(u_*)$ a.e. in Ω. Moreover, the second convergence implies also that (u_n) is bounded in $L^q(\Omega)$ for $q \in [1, 2^*)$ and, since $|F(u_n)| \leq a\,|u_n|$, the Dominated Convergence Theorem (DCT) can be applied to give us

$$\int_\Omega F(u_n)\,dx \longrightarrow \int_\Omega F(u)\,dx.$$

At last, by the weak lower semi-continuity of the norm,

$$\|u\|^2 \leq \liminf_{n\to\infty} \|u_n\|^2.$$

Thus, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(u)\,dx \\ &\leq \frac{1}{2}\liminf_{n\to\infty}\int_\Omega |\nabla u_n|^2\,dx - \lim_{n\to\infty}\int_\Omega F(u_n)\,dx \\ &= \liminf_{n\to\infty}\left(\frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx - \int_\Omega F(u_n)\,dx\right) \\ &= \liminf_{n\to\infty} I(u_n) = m. \end{aligned} \tag{1.5}$$

With this, we see it can only be $I(u) = m = \inf_{v\in H_0^1(\Omega)} I(v)$. We have obtained, thus, that u is a global minimum for I, that is, a critical point for this functional. $\square$

Remark 1.3.2. One thing we may want to guarantee is that our attained solution is different from the trivial solution, that is, the constant function $u \equiv 0$. In the terms of Theorem 1.3.1, this cannot be proven, meaning we need additional conditions over f. If, for example, f behaves as expressed in (h_1) only for big enough arguments and is continuous close to the origin, but such that $f(0) \neq 0$, then we obviously obtain that the trivial function is not a solution to Problem (1.1). Additionally, we may ask (see [4]) that f also satisfies the following condition

(h_1') $\liminf_{t \to 0^+} \frac{f(t)}{t} > \lambda_1$,

where λ_1 is the first eigenvalue to the $-\Delta$ operator.

Now, if we examine carefully our proof, we see that two factors in the assumptions were crucial: first, we needed I to be a coercive and bounded from below, which was given here by the assumption that f is a bounded function; second, the fact that we were able to bound the function $|F(u_n)|$ by a multiple of $|u_n|$ gave us the possibility of using the DCT to conclude the continuity of the second term of I (and thus the semi-continuity of I itself). What we can see now is that if the $\int_\Omega F(u)\, dx$ were to be bounded by any term with a growth below the quadratic growth in the norm $\|u\|$, then I would still be coercive. Meanwhile, considering f to be below the so called critical growth, that is,

$$|f(s)| \leq a + b\,|s|^p, \quad \forall\, s \in \mathbb{R},$$

$p \in [0, 2^* - 1)$, we shall have

$$|F(s)| \leq a_1 + b_1\,|s|^{p+1}, \quad \forall\, s \in \mathbb{R},$$

and the Sobolev Embedding will again provide $|F(u_n)|$ uniformly bounded by a function $w \in L^1(\Omega)$. The use of the DCT would then still be possible and I would remain weakly lower semi-continuous.

Therefore, we can only, for the time being, assume

(h_2) The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies:

$$\text{There exists } a, b > 0 \text{ such that } |f(s)| \leq a + b\,|s|^p\ , \quad \forall\, s \in \mathbb{R},$$

where $p \in (0, 1)$.

With this, the two main factors in Theorem 1.3.1 are preserved and it can thus be proved, in the same spirit, the following.

Theorem 1.3.2. *If f is a function satisfying (h_2), then there exists a solution to Problem (1.1).*

Remark 1.3.3. If b is small enough, we are able to allow f to achieve linear growth, that is, to take $p = 1$ in (h_2), and still obtain the term $\frac{1}{2}\|u\|^2$ dominating at infinity. More specifically, we must require $b < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue for the Laplacian operator. See, for example, [4, Theorem 2.1.6].

Remark 1.3.4. Note that the coercive property is a characterization of the behavior of I at infinity, implying that the growth condition for the function f near the origin does not matter so much. We can, therefore, assume weaker conditions, such as

(h_1') The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies:

$$\limsup_{s\to\pm\infty} \frac{|f(s)|}{|s|} < +\infty.$$

When f is above the linear growth, called *superlinear case*, we do not have a lower bound any more and this, obviously, hinders the use of minimization methods. Variations must be added then to suit each case. In each of then, it is still important how the growth condition of f is related to the critical Sobolev exponent $2^* = \frac{2N}{N-2}$, since it is always used throughout our proofs the Sobolev Embedding Theorems. As a first example, we note that, if the term $\int_\Omega F(u)\,dx$ in (1.2) were negative, we do not need to bound it by a power of the norm $\|u\|$ for I to be coercive. A possible condition is this

(h_3) The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies:

$$\text{There exists } a, b > 0 \text{ such that } |f(s)| \le a + b\,|s|^{2^*-1}\,, \quad \forall\, s \in \mathbb{R}.$$

Moreover,

$$f(s)s \le 0\,, \quad \forall\, s \in \mathbb{R}.$$

With this, it can be proven that F satisfies

$$|F(s)| \le a_1 + b_1\,|s|^{2^*}\,, \quad \forall\, s \in \mathbb{R},$$

and also $F(s) \le 0$, $\forall\, s \in \mathbb{R}$.

Thus, we can again adapt the proof of Theorem 1.3.1 and obtain

Theorem 1.3.3. *If f is a function satisfying (h_3), then there exists a solution to Problem (1.1).*

Proof. See [4, Theorem 2.1.11]. □

The second example is perhaps the most simple one in characterization, but is one which does possesses serious problems, where f is the power function $f(s) = |s|^{p-2}\,s$, with $p \in (2, 2^*)$. The problem is then

$$\begin{cases} -\Delta u = |u|^{p-2}\,u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.6}$$

For this problem, the functional needed is

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \; dx - \frac{1}{p}\int_\Omega |u|^p \; dx = \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p \tag{1.7}$$

and we can see directly how the coercitivity is lost for I if we write $I(tu)$,

$$\lim_{t\to\infty} I(tu) = \lim_{t\to\infty}\left(\frac{t^2}{2}\|u\|^2 - \frac{t^p}{p}|u|_p^p\right) = -\infty,$$

since $p > 2$.

What can be done to overcome this problem is to constrain the functional I to a subset of $H_0^1(\Omega)$ which recovers this property. Going through the literature (see for example, [4, 5, 6], and references therein), we find that possible such subsets are:

1. The sphere of $L^p(\Omega)$ in $H_0^1(\Omega)$,

   $$\Sigma_\beta = \{u \in H_0^1(\Omega) \; ; \; |u|_p^p = \beta\}.$$

   With this, the functional I restricted to Σ_β becomes

   $$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\beta,$$

   which is obviously coercive and bounded below.

   If we apply our proof for the attainment of a solution, however, we shall find a function that satisfies the equation for a weak solution for (1.6),

   $$\int_\Omega \nabla u \nabla v \; dx = \int_\Omega |u|^p \, v \; dx,$$

   only for test functions $v \in T_u\Sigma_\beta$, the tangent space of Σ at u. What is left for us to do is to prove that such u will satisfy this same equation for any $v \in H_0^1(\Omega)$.

2. The *Nehari Manifold* $\mathcal{N} \subset H_0^1(\Omega)$,

   $$\mathcal{N} = \{u \in H_0^1(\Omega) \; ; \; u \not\equiv 0 \; , \; I'(u)u = 0\}.$$

   We can see that condition defining this subset is equivalent, given the characterization of I in (1.7), to the following

   $$\|u\|^2 = \int_\Omega |\nabla u|^2 \; dx = \int_\Omega |u|^p \; dx = |u|_p^p.$$

   Now, the functional I restricted to $\mathcal{N}$ will be

   $$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\|u\|^2 = (\frac{1}{2} - \frac{1}{p})\|u\|^2,$$

   which is, once more, coercive and bounded below. One more time, we would need first to prove that there exists a minimizing function of I restricted to $\mathcal{N}$ and, after that, extend its condition as a weak solution to the entire space $H_0^1(\Omega)$.

Furthermore, we shall treat in this work problems with another type of criticality in the nonlinearity. This one is related to the Sobolev Embedding Theorems when $kp = N$. This case of the embedding is important when we deal with problems containing the N-Laplacian as the operator. Therefore, let us consider the following elliptic problem

$$\begin{cases} -\Delta_N u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.8}$$

where Δ_N represents the operator

$$\Delta_N u = div\ (|\nabla u|^{N-2}\, \nabla u).$$

When u is regular enough, we can write it as

$$\Delta_N u = |\nabla u|^{N-2}\, \Delta u + (N-2)\, |\nabla u|^{N-4} \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

Through another point of view, we can see the N-Laplacian as an operator from $W_0^{1,N}(\Omega)$ to its dual, given by

$$\langle \Delta_N u, v\rangle = -\int_\Omega |\nabla u|^{N-2} \nabla u \nabla v\ dx\ , \quad \forall\ u, v \in W_0^{1,N}(\Omega). \tag{1.9}$$

With this, we can see that, if u is a weak solution to (1.8), we have, using u as the test function,

$$\int_\Omega |\nabla u|^N\ dx = \int_\Omega f(u) u\ dx,$$

that means,

$$\|u\|^N_{W_0^{1,N}(\Omega)} - \int_\Omega f(u) u\ dx = 0.$$

Therefore, the functional associated to this problem,

$$I(u) = \frac{1}{2}\, \|u\|^N_{W_0^{1,N}(\Omega)} - \int_\Omega F(u)\ dx,$$

will be coercive and bounded below if f satisfies the growth condition

$$|f(s)| \leq a + b\, |s|^q\ , \text{ for } 0 < q < N-1.$$

Furthermore, since the embedding $W_0^{1,N}(\Omega) \hookrightarrow L^s(\Omega)$ is compact for every $s \in [1, +\infty)$, we have no problem extending our proofs involving constrained minimization for the N-Laplacian case here (see [4]). What is then the critical condition that can be studied here? The answer comes from N. Trudinger [7] and J. Moser [8]. The first, using the power series expansion of the exponential function and some Sobolev estimates, was able to prove that for any $u \in W_0^{1,N}(\Omega)$, we can bound the integral

$$\int_\Omega e^{\alpha |u|^{N'}}\ dx,$$

where $N' = \frac{N}{N-1}$ and α is a positive constant, independent of u.

Following this, Moser was able to improve this result, concluding that there exists a $\alpha = \alpha_N$ such that the above estimate is valid for $\alpha \le \alpha_N$ and, moreover, is false for $\alpha > \alpha_N$. More specifically, the exact result achieved was that

$$\sup_{\|u\|^p_{W_0^{1,N}(\Omega)} \le 1} \int_\Omega e^{\alpha|u|^{N'}}\, dx \quad \begin{cases} \le c\,|\Omega|\,, & \text{if } \alpha \le \alpha_N, \\ = +\infty, & \text{if } \alpha > \alpha_N, \end{cases} \tag{1.10}$$

for some constant $c > 0$ dependent on N, where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, being ω_{N-1} the measure of the unit sphere in $\mathbb{R}^N$. Thus, inequality (1.10) is called the Trudinger-Moser inequality and the character of α renders α_N the name *critical Trudinger-Moser growth.* This is, therefore, the growth case which divides the problems in terms of difficulty.

1.4 FURTHER CHAPTERS

Now, we shall present, briefly, the problems we shall study in each following chapter. More context will be given for all of them at the right moment. For now, we only cite the main characteristics of the problems and the developments achieved up to now by the current literature. All the main theorems present in this section and proved in the following chapters were fitted into articles and submitted to esteemed journals, a fact which reiterates their importance and contemporaneity.

In the second chapter, we look for solutions to the following class of elliptic nonlinear problems

$$\begin{cases} -\Delta u = \lambda u^{q-1} + f(u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ B(u) = 0, & x \in \partial\Omega. \end{cases} \tag{1.11}$$

For this particular problem, we will assume f to be of supercritical growth, in the sense of Sobolev, and the main difference from the other cases is the assumption of the B operator as the boundary condition, characterizing what we call a mixed Dirichlet-Neumann boundary conditions. More precisely, B is defined as

$$B(u) = u\chi_{\Sigma_1} + \frac{\partial u}{\partial \nu}\chi_{\Sigma_2}, \tag{BC}$$

where both Σ_1, Σ_2 are smooth (N-1)-dimensional sub-manifolds of $\partial\Omega$ with positive measure and such that $\overline{\Sigma_1} \cup \overline{\Sigma_2} = \partial\Omega$, $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\overline{\Sigma_1} \cap \overline{\Sigma_2} = \Gamma$ is a smooth (N-2)-dimensional sub-manifold. Here, ν is the outward unitary normal vector to the boundary $\partial\Omega$ and χ_A is the characteristic function of the set A.

Remark 1.4.1. The nomenclature “mixed Dirichlet-Neumann boundary conditions” should be readily understood here, since the equality $B(u) = 0$ requires that u vanishes at some part of $\partial\Omega$, namely the sub-manifold Σ_1, which constitutes the condition imposed by

Dirichlet problems, and that the exterior derivative (or normal derivative) vanishes in the complementary subset of the boundary $\partial\Omega$, condition asked by Neumann problems.

We have seen how results become more scarce when we talk about nonlinearities above the linear growth, with authors having to substitute the form of a power function u^p for some other conditions restricting the behavior of f. One great contribution not mentioned above was done by the work of Ambrosetti and Rabinowitz [9], where it was assumed the problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.12}$$

with the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{array}{ll} (I_1) & f(x,0) = 0\ , \quad \lim_{s\to 0} \frac{f(x,s)}{s} = 0 \\ (I_2) & |f(x,s)| \leq a_1 + a_2|s|^p \quad , \quad a_1, a_2 > 0\ , \quad 1 < p < \frac{N+2}{N-2} \end{array}$$

together with the following condition

(AR) There exists $r > 0$ and $\theta > 2$ such that
$0 < \theta\ F(x,s) \leq f(x,s)$, for $s > r$, where $F(x,s) = \int_0^s f(x,t)dt$.

This last condition is known as the Ambrosetti-Rabinowitz (AR) condition and is crucial to ensuring that the related functional still possesses the compactness required for the Mountain Pass Theorem, as presented by the authors in the latter paper.

Seen that the (AR) condition is yet a limiting factor to the more general results, many papers have then tried to drop this assumption using different techniques. In [10] or [11], for example, the authors used a weaker version of (AR), concerning the growth condition of $F(x,t)$,

$$\lim_{s\to\infty} \frac{F(x,s)}{s^2} = +\infty$$

to prove the existence of a nontrivial weak solution to problem (1.12) for functions f satisfying the subcritical growth (I_2) and that are almost linear near to the origin, using again the Mountain Pass Theorem and adding also a parameter $\lambda > 0$ multiplying f.

Moreover, we can give more references to supercritical problems which achieved the goal of generalizing the results beyond the (AR) condition. In [12], it was considered the following problem

$$\begin{cases} -\Delta u = \lambda u^{q(r)-1} + f(r,u), & x \in B, \\ u > 0, & x \in B, \\ u = 0, & x \in \partial B, \end{cases} \tag{1.13}$$

where $B \subset \mathbb{R}^N$ is the open unit ball and with f depending on the radial coordinate $r = |x|$ and satisfying a variable exponent growth

$$0 \leq sf(r,s) \leq a_1|s|^{p(r)},$$

the function $p(r) = 2^* + \alpha r$, $\alpha > 0$. There, the authors already generalized the problem treated in [13], for which $\lambda = 0$ and where f was simply $|s|^{p(r)-2}s$. Additionally, in [14], the authors treated a similar radial problem, considering the whole $\mathbb{R}^N$ and again without the need for the (AR) condition.

On the other hand, another aspect of the problem we can give rise to more general results is the boundary conditions. Problems with mixed boundary conditions have shown to be more and more important in recent years, as exemplified in [15] and references therein. The great work done by Peral and Colorado ([16]) treats the subcritical problem subjected to the mixed Dirichlet-Neumann boundary condition given by (BC).

$$\begin{cases} -\Delta u = \lambda u^{q-1} + u^r, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ B(u) = 0, & x \in \partial\Omega, \end{cases} \tag{1.14}$$

where $1 < r < 2^* - 1$, $0 < q < r$, $\lambda > 0$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain.

Furthermore, still in this paper, the authors not only prove the existence of a solution to problem (1.14), but also achieve multiplicity for such solutions and a non-existence result for the problem depending on the parameter λ, as well as on the q parameter, obtaining different results for the sublinear perturbation case ($q < 1$) and the eigenvalue case ($q = 1$). It is worth citing also the results obtained in estimating the L^∞ norm of solutions for (1.14). We observe, however, that the nonlinearity of equation (1.14) is still quite specific, not achieving the critical growth or considering functions beyond the polynomial function.

Paper [16], however, was not the first one to deal with the change in the boundary conditions. Grossi, in the work [17], proved the existence for a version of problem (1.14), with $q = 1$ and assuming the critical growth $r = 2^* - 1$. Adimurthi, Pacella and Yadava, in [18] treat the mixed Dirichlet-Neumann problem (1.14) for the equation $-\Delta u + \lambda \alpha u = u^{2^*-1}$, where $\alpha \in C^1(\overline{\Omega})$, assuming some specific geometric conditions for the Σ_2 component of $\partial\Omega$. We see that the consideration of the critical Sobolev growth leads to significant scarcer results.

Nevertheless, following the development of the field, Ding and Tang, in [15], studied a Hardy-Sobolev critical singular problem, also with mixed Dirichlet-Neumann boundary conditions, which generalizes problem (1.14) to the critical Sobolev exponent case and also achieve other interesting results of multiplicity and non-existence concerning the case with Hardy terms. We intend to generalize the results even further by treating the case of a superlinear nonlinearity (and without restraining ourselves to the polynomial function), while still considering mixed Dirichlet-Neumann conditions at the boundary.

Turning back to Problem (1.11), the mixed boundary condition forces us to abandon the Sobolev Space $H_0^1(\Omega)$ for a more suitable one, namely

$$E_{\Sigma_1}(\Omega) := \left\{v \in H^1(\Omega); v = 0 \text{ on } \Sigma_1\right\},$$

which does provides the correct boundary information for a solution of (1.11). It can be seen that it preserves the properties of $H_0^1(\Omega)$ which are important for our solution, like the structure of a Hilbert space and the Sobolev Embedding Theorems on $L^p(\Omega)$ spaces. Furthermore, the presence of the mixed operator B compels us to adapt certain results proper to Dirichlet Boundary conditions only, such as comparison results.

Over the nonlinearity f, we first assume the basic condition that its image be positive for a positive argument, namely the sign property:

(H_1) It has the sign property, namely:

$$0 \leq f(t)t \quad , \quad t \in \mathbb{R} \, ;$$

Furthermore, as we have mentioned, f is made supercritical in terms of Sobolev growth. With this, we mean that we have

(H_2) It has a critical or supercritical growth at infinite, in the sense that

$$\liminf_{t\to\infty} \frac{f(t)}{t^r} = \infty \quad , \qquad \forall \quad r \in \left(1, \frac{N+2}{N-2}\right) \; ;$$

We make, however, the following additional assumptions over the growth condition of f

(H_3) We assume that there exists a number $\theta > 0$ such that

$$\limsup_{t\to\infty} \frac{f(t)}{t^{2^*-1+\theta}} < \infty \, ;$$

(H_4) There exists a sequence (M_n) with $M_n \to \infty$ and such that, for every $r \in (0, \frac{N+2}{N-1})$,

$$t \in [0, M_n] \;\Rightarrow\; \frac{f(t)}{t^r} \leq \frac{f(M_n)}{(M_n)^r}.$$

This last light condition on the increasing behavior of every $\frac{f(t)}{t^r}$ is what helps us overcome the supercritical growth of f. We prove the existence and positiveness of a weak solution for (1.11), fact summarized by the following theorem,

Theorem 1.4.1. *If $f : [0,\infty) \longrightarrow \mathbb{R}$ is a continuous function satisfying the growth conditions (H_1) - (H_4), then there exists $\gamma > 0$ and $\Lambda > 0$ such that problem* (1.11) *has a weak solution $u_\lambda \in E_{\Sigma_1}(\Omega) \cap W^{2,\frac{2^*}{2^*-1}}(\Omega)$ whenever $0 < \theta < \gamma$ and $0 < \lambda < \Lambda$.*

Furthermore, we may also prove, for the same problem, a nonexistence result concerning the range of the parameter λ. Namely that the set of parameters λ for which (1.11) has a solution is bounded above. More precisely, we prove

Theorem 1.4.2. *If f is a continuous function satisfying the growth conditions (H_1) - (H_3), then the set of parameters λ for which problem* (1.11) *has a solution is bounded from above.*

Both Theorems 1.4.1 and 1.4.2 are present in our submitted and published article [19], in the journal *Complex Variables and Elliptic Equations.*

Following that, we devote the third chapter to the resolution of the system, named Schrodinger-Poisson (SP) type system, given by

$$\begin{cases} -\Delta u - \phi u^{2^*-2} = \frac{\lambda}{u^\gamma} & \text{in} \quad \Omega, \\ -\Delta \phi = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.15}$$

Apart from the evident difference of presenting two equations instead of one, like in (1.11), this problem differs from the first by introducing a singularity in one of the nonlinear terms, while restraining the second one, characterized by the function f, to a subcritical or critical growth, with no restriction on the behavior of f. This is an important improvement since the critical growth is a quite important hindering factor to more classical methods such as variational ones, as we have just seen. We ask that f satisfies only

$$0 \leq f(s)s \leq L|s|^{2^*}, \qquad L > 0. \tag{1.16}$$

Problem (1.15) was shown to have a quite rich history in the area of study of mathematical physics. One of the most general definitions of a Schrödinger-Poisson system can be expressed by the following coupled equations

$$\begin{cases} -\Delta u + V(x)u + k(x)\phi|u|^{q-2}u = f(x,u) & \text{in} \quad \mathbb{R}^3, \\ -\Delta \phi = k(x)|u|^q & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{1.17}$$

Benci and Fortunato [20] introduced the study of such a system to represent the physical model of a charged particle interacting with an electromagnetic field, in the quantum mechanics formulation, when assumed to have a stationary solution (for more details on the applications of such study, see [21, 22] and references therein). There, it took the following eigenvalue formulation

$$\begin{cases} -\frac{1}{2}\Delta u - \phi u = \omega u & \text{in} \quad \Omega, \\ \Delta \phi = 4\pi u^2 & \text{in} \quad \Omega, \\ \phi = g & \text{on} \quad \partial\Omega. \end{cases} \tag{1.18}$$

The condition $\phi = g$ represents our setting of the potential in the boundary of a given subset Ω of $\mathbb{R}^3$ and g is assumed to be continuous. In addition, since u represents the amplitude of the wave function of the contained particle, it is necessary to impose the condition of normalization on Ω, namely

$$\int_\Omega u^2\, dx = 1.$$

We cite that the name of such system is due to the presence of a nonlinear stationary Schrödinger equation, one of the central pieces of quantum mechanics, coupled with a Poisson equation, which is derived from the Maxwell equations for the electric potential (for this reason the two equations can also be referred as Schrödinger-Maxwell system).

For this reason, many papers have been interested in finding results about the existence and multiplicity of solutions to Schrödinger-Poisson type systems, specially when it includes terms with critical growth, where the methods available become more scanty. For bounded domains, there was, for a long period, a certain scarcity of results of existence for systems such as (1.17) and the ones attained still had a certain degree of restriction. Nonetheless, there has been, more recently, a number of new results advancing the findings for this type of problem.

In [23], it was considered the system

$$\begin{cases} -\Delta u = \lambda u + \phi |u|^{2^*-3}u & \text{in} \quad \Omega, \\ -\Delta \phi = |u|^{2^*-1} & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.19}$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 3$. Despite the choice of a critical exponent in the second equation, the authors showed that, by using a reduction method, problem (1.19) can still be treated through variational theory. Not only do they obtain solution for suitable values of λ, but the also derive some nonexistence results for particular values of the same parameter. This work already serves as a generalization for [24], where it is consider the same system (1.19), with $\Omega = B_R$ and $N = 3$.

Meanwhile, paper [25] treated the problem of a generalized Schrödinger-Poisson type system

$$\begin{cases} -\Delta u + \epsilon q \phi f(u) = |u|^{p-1}u & \text{in} \quad \Omega, \\ -\Delta \phi = 2qF(u) & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.20}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $1 < p < 5$, $\epsilon = \pm 1$, $q > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $F(s) = \int_0^s f(t)dt$ denotes its primitive function. This generalizes various papers preceding it by considering f different from the identity function. It reproduces, for example, the findings of [20] if $p = 1$, where the system was treated as an eigenvalue problem. The existence results, however, depend strongly on the assumption of small value for the parameter q, meaning that it still restricts more general cases. Again the assumptions therein allow the use of variational methods. Moreover, they were also able to achieve multiplicity and non-existence results for certain cases.

Other papers introduced to this system a singular term in the first equation and studied the modifications needed for such a case. In [26], for example, the author studied the problem

$$\begin{cases} -\Delta u + \eta\phi u = \lambda u^{-r} & \text{in} \quad \Omega, \\ -\Delta\phi = u^2 & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.21}$$

finding different results of existence when varying the domain of the parameter λ and for $\eta = \pm 1$. The quite specific form of the nonlinearity in the second equation and its quadratic form allowed for quite promising conclusions. More specifically, it was presented the following results

Theorem 1.4.3. *Assume $\eta = 1$. Then system* (1.21) *has a unique positive solution for every $\lambda > 0$.*

Theorem 1.4.4. *Assume $\eta = -1$. Then there exists a constant $\Lambda = \Lambda(r, \Omega) > 0$, such that for any $\lambda \in (0, \Lambda)$ system* (1.21) *has at least two different positive solutions.*

Finally, in [27], the authors introduced the critical growth to this system already containing a singularity, treating the following problem

$$\begin{cases} -\Delta u + \eta\phi u^{2^*-2} = \frac{\lambda}{u^\gamma} & \text{in} \quad \Omega, \\ -\Delta\phi = u^{2^*-1} & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.22}$$

where $\eta = \pm 1$, $\gamma \in (0, 1)$ is a constant and $\lambda > 0$ again a real parameter. They obtained the same results, obtaining in both cases existence and uniqueness (or multiplicity) based on variational methods.

It is used here a sequence of auxiliary functions f_k called Strauss approximation, in virtue of W. A. Strauss, which introduced them in [28]. They are important elements to our work, since they are regular enough - Lipschitz and bounded functions - and approximate f uniformly in bounded domains.

For problem (1.15), we have achieved the following result, which is part of the article [29], recently published in the *Journal of Mathematical Analysis and Applications*.

Theorem 1.4.5. *If $f : [0, \infty) \longrightarrow \mathbb{R}$ is a continuous function satisfying the growth condition* (1.16)*. Then there exists $\Lambda > 0$ such that, for every $0 < \lambda < \Lambda$, Problem* (1.15) *has a pair of solutions $u_\lambda, \phi_\lambda \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$, with $p = \frac{2^*}{2^*-1}$.*

Finally, the last problem we deal with, treated in Chapter 4, involves the more general N-Laplacian operator, defined in (1.9). We study the following system, again of Schrodinger-Poisson type,

$$\begin{cases} -\Delta_N u - \phi \frac{f(u)}{u} = \frac{\lambda}{u^\gamma} & \text{in} \quad \Omega, \\ -\Delta_N \phi = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.23}$$

The main difference to (1.15), apart from - but related to - the N-Laplacian operator, is that we impose now a exponential growth on f, dictated by the Trudinger-Moser inequality (1.10). We ask f to satisfy

$$0 \leq f(s)s \leq L|s|^{r+1} \exp\{\alpha s^{\frac{N}{N-1}}\}, \qquad L, \alpha > 0, \quad r > N-1. \tag{1.24}$$

Also, as we have already seen, the change in operator forces us to work in the $W_0^{1,N}(\Omega)$ and some results have again to be adapted to this new frame of work.

It is interesting to see how this condition poses a critical growth in the absence of the critical exponent in the Sobolev sense, as is the case for $\mathbb{R}^N$, where $N = 2$. A good example of this condition being applied to a (SP) system is given by [30], where the nonlinearity f is present in the first equation,

$$\begin{cases} -\Delta u + \phi u = f(u) & \text{in} \quad \Omega, \\ \Delta \phi = 2\pi u^2 & \text{in} \quad \Omega. \end{cases} \tag{1.25}$$

There, they treat the “zero mass case", translated as the condition $\frac{f(u)}{u} \to 0$ as $u \to 0$. As we said, f is supposed to obey a critical exponential growth, satisfying

(F_1) There exists a constant $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha t^2}} = 0 \quad , \qquad \forall\, \alpha > \alpha_0$$

and

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha t^2}} = +\infty \quad , \qquad \forall\, \alpha \leq \alpha_0.$$

For the existence of a ground state solution to (1.25), more conditions have to be assumed, including some which dictate the behavior of f, such as

(F_2) There exists $l \in [0, +\infty)$ such that

$$\lim_{t\to 0} \frac{f(t)}{|t|t} = l;$$

(F_3) For all $t \in \mathbb{R}$,

$$\frac{1}{3} f(t)t \geq F(t) = \int_0^t f(s)\, ds \geq 0;$$

We see then that the papers we have mentioned which treats (SP) type systems in the case $N \geq 3$ had to impart a subcritical or critical growth, in the sense of Sobolev, to the nonlinearity, with results in the critical case already being scarce due to the lack of compactness of the spaces we encounter, which hinders the use of variational methods. The case $N = 2$ and critical exponential growth encounters different but also important complications, which leads to several additional constraints, as we have just seen. Moreover, the presence of a singularity produces even more obstacles, now being the possibility of a blowup at certain points in Ω. The advantages of this work are then better seen in the conditions imposed over the nonlinearity. For the $\eta = -1$ case, we make use of the non-variational Galerkin Method, which helps us expand the reach of our results to more general conditions over f, while we can treat growth conditions beyond the critical Sobolev growth (for that we also depend greatly on the Trudinger-Moser inequality). At the same time, it brings us no insurmountable difficulties in dealing with the singular term. We do this while keeping the nonlinearity f quite general in its behavior. Nonetheless, we could not use the same approach for the case $\eta = 1$. For this purpose, we overcome these difficulties by combining suitable estimates and Schauder fixed point theory and we find the existence of solutions.

We now state our main results.

Theorem 1.4.6. *If $\eta = -1$ and $f : [0, \infty) \longrightarrow \mathbb{R}$ is a continuous function satisfying the growth condition* (1.24). *Then there exists $\Lambda > 0$ such that, for every $0 < \lambda < \Lambda$, problem* (1.23) *has a solution pair $u_\lambda, \phi_\lambda \in W_0^{1,N}(\Omega)$.*

Before continuing, we would like to point out a remark that will be useful later one.

Remark 1.4.2. When f is such that

$$|f(s)| \leq c_1 + c_2|s|^{N-1}, \ \forall\, s \in \mathbb{R},$$

then the solution pair u_λ, ϕ_λ obtained in Theorem 1.4.6 will belong to $C^{1,\tau}(\overline{\Omega})$, for some $\tau \in (0, 1)$. This follows directly from the considerations in [31] (we may notice that a more general nonlinearity containing a singularity is used as a prototype).

When $\eta = 1$, we consider the limit problem:

$$\begin{cases} -\Delta_N u + \phi u^{r-1} \exp\{\alpha u^{N'}\} = \frac{\lambda}{u^\gamma} & \text{in} \quad \Omega, \\ -\Delta_N \phi = u^r \exp\{\alpha u^{N'}\} & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.26}$$

with $\alpha \geq 0, \lambda > 0$ being again real parameters, as is $\gamma \in (0, 1)$, and recalling that $N' = \frac{N}{N-1}$. For this case, our result is as follows.

Theorem 1.4.7. *Let us suppose $\alpha > 0$ arbitrary and r such that*

$$(\gamma + rN' - 1)\left(\frac{1-\gamma}{N-1}\right) > 1.$$

Then there exists $\Lambda > 0$ such that, for every $0 < \lambda < \Lambda$, problem (P_4) has a solution pair $u_\lambda, \phi_\lambda \in C_0^1(\overline{\Omega})$. If $\alpha = 0$ the problem (P_4) has a unique positive solution for every $0 < \lambda < \Lambda$.

In resemblance to the first results, Theorems 1.4.6 and 1.4.7 are both two results of the article [], submitted to the .

2 AN EQUATION INVOLVING SUPERCRITICAL SOBOLEV GROWTH WITH MIXED DIRICHLET-NEUMANN BOUNDARY CONDITIONS

In this Chapter, we look at our first elliptic problem resolved by means of the Galerkin Method. We shall consider, as expressed in Chapter 1, the following elliptic problem

$$\begin{cases} -\Delta u = \lambda u^{q-1} + f(u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ B(u) = 0, & x \in \partial\Omega, \end{cases} \tag{P_1}$$

where the parameters satisfy $1 < q < 2$ and $\lambda > 0$. The first term is thus a sublinear perturbation and is present to avoid non existence results such as Pohozaev's. Furthermore, we will consider here f to be a continuous function satisfying the following conditions:

(H_1) It has the sign property, namely:

$$0 \leq f(t)t\ , \quad t \in \mathbb{R}\ ;$$

(H_2) It has a critical or supercritical growth at infinite, in the sense that

$$\liminf_{t\to\infty} \frac{f(t)}{t^r} = +\infty\ , \qquad \forall\, r \in (1, 2^* - 1]\ \ ;$$

(H_3) We assume that there exists a number $\theta > 0$ such that

$$\limsup_{t\to\infty} \frac{f(t)}{t^{2^*-1+\theta}} < +\infty\ ;$$

(H_4) At last, we assume that there exists a sequence (M_n) with $M_n \to \infty$ and such that, for every $r \in (0, 2^* - 1)$,

$$t \in [0, M_n] \implies \frac{f(t)}{t^r} \leq \frac{f(M_n)}{(M_n)^r}.$$

Additionally, we also recall from Section 1.4 that the boundary condition is given by $B(u)$ defined as follows

$$B(u) = u\chi_{\Sigma_1} + \frac{\partial u}{\partial \nu}\chi_{\Sigma_2}. \tag{BC}$$

Remark 2.0.1. In sum, conditions (H_1) - (H_4) can all be (loosely) summed up by the following: We ask f to be positive when its argument is positive (which is what we are looking for, since we ask $u > 0$) and of supercritical growth at infinity, but we assume that the behavior of this "supercriticality" be ever growing, at last in supremum.

We present again our main results for the present chapter, for the convenience of the reader.

Theorem 2.0.1. *If $f : [0,\infty) \longrightarrow \mathbb{R}$ is a continuous function satisfying the growth conditions (H_1) - (H_4), then there exists $\gamma > 0$ and $\Lambda > 0$ such that problem* (P_1) *has a weak solution $u_\lambda \in E_{\Sigma_1}(\Omega) \cap W^{2,p}(\Omega)$, $p = \frac{2^*}{2^*-1}$, whenever $0 < \theta < \gamma$ and $0 < \lambda < \Lambda$.*

Following this existence result, we shall prove a non-existence one, concerning the nature of the set in which λ must be so that Problem (P_1) has a solution. More specifically, we prove

Theorem 2.0.2. *If f is a continuous function satisfying the growth conditions (H_1) - (H_3), then the set of parameters λ for which problem* (1.11) *has a solution is bounded from above.*

As we have pointed out, Theorems 2.0.1 and 2.0.2 comprise a larger number of functions than those considered in the current literature. For the sake of exemplification, we would like to mention the following functions which verify conditions (H_1) - (H_4), but do not verify, for example, the (AR) condition. They are part of a much greater set of functions addressed by the results of the present chapter.

$$i)\;\; f(u) = \begin{cases} 0, & \text{if} \quad u \leq 0, \\ u^{2^*-1+\theta}(ln(u))_+, & \text{if} \quad u \geq 0, \end{cases}$$

$$ii)\;\; f(u) = \begin{cases} 0, & \text{if} \quad u \leq 0, \\ u^{2^*-1+\theta}\; sen^2(u), & \text{if} \quad u \geq 0. \end{cases}$$

We have also mentioned in Chapter 1 that it is not enough to work with the space $H^1(\Omega)$ and is not sufficient to work with $H_0^1(\Omega)$ either, since we need our solution to be zero at some, but not at all, parts of the boundary $\partial\Omega$. The best choice then is to work with the space $E_{\Sigma_1}(\Omega) := \{v \in H^1(\Omega); v = 0 \text{ on } \Sigma_1\}$, which can also be identified as the closure of $C_c^1(\Omega \cup \Sigma_2)$ with the norm of $H^1(\Omega)$ (in parallel to the characterization of $H_0^1(\Omega)$ as the closure of $C_c^1(\Omega)$ with the same norm). Its effectiveness relies on the fact that we still have the properties that make $H_0^1(\Omega)$ suitable to problems with Dirichlet boundary conditions, such as the continuous embedding (see, for more, [17])

$$E_{\Sigma_1}(\Omega) \hookrightarrow L^q(\Omega)\,, \quad q \in [1, 2^*]\,, \tag{2.1}$$

or the fact that $E_{\Sigma_1}(\Omega)$ is a separable Hilbert space, which is a crucial demand because it implies that this space has an orthonormal basis, allowing us to use the Galerkin Method. The norm of this space is initially defined as the norm on $H^1(\Omega)$ but, as Σ_1, Σ_2 are assumed to be of positive measure, it can be shown (see [32]) that the Poincaré Inequality is satisfied in E_{Σ_1}, so that one can use the equivalent norm

$$\|u\|_{E_{\Sigma_1}} = \left(\int_\Omega |\nabla u|^2 \, dx\right)^{1/2},$$

which we shall denote from now on, throughout this chapter, only by $\|.\|$, for convenience.

The ideas that permit us to consider the case of a supercritical nonlinearity were inspired and adapted from the work due to Alves and de Figueiredo [33] and it was required a significant amount of modifications on the results used to adjust the findings to our studied problem. Since we are looking for positive solutions, we can assume $f(s) = 0$ if $s \in (-\infty, 0)$.

Before the proper beginning to our proof, we first give a comparison theorem, which we have adapted from the Dirichlet boundary case (result [34, Lemma 3.3.]) to our mixed-boundary problem. Consider the problem

$$\begin{cases} -\Delta u = g(u), & \text{in} \quad \Omega, \\ u > 0, & \text{in} \quad \Omega, \\ B(u) = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

with $g(u) \geq 0$ for $u > 0$. A weak supersolution of a Dirichlet-Neumann boundary problem such as (2.2) is defined as a function $u \in E_{\Sigma_1}(\Omega)$ such that $u > 0$ in Ω and such that

$$\int_\Omega \nabla u \nabla \phi \, dx \geq \int_\Omega g(u)\phi \, dx$$

for every test non-negative function $\phi \in E_{\Sigma_1}(\Omega)$. A weak subsolution for (2.2) is defined in the same manner with the inequality switched. It is evident that a weak solution is both a weak supersolution and a weak subsolution.

Theorem 2.0.3. *If $u, v \in E_{\Sigma_1}(\Omega)$ are, respectively, a weak supersolution and a weak subsolution to problem (2.2), with g satisfying $g(s) \geq 0$ for $s \geq 0$ and $g(s)/s$ is a decreasing function, then $u \geq v$ a.e. in Ω.*

Proof. First, let $\theta(t)$ be a smooth non-decreasing function such that $\theta(t) = 0$ for $t \leq 0$ and $\theta(1) = 1$ for $t \geq 1$. Moreover, for $\epsilon > 0$, set

$$\theta_\epsilon(t) = \theta\left(\frac{t}{\epsilon}\right).$$

By hypothesis, we must have

$$\int_\Omega \nabla u \nabla(v\theta_\epsilon(v-u)) \, dx \geq \int_\Omega g(u)v\theta_\epsilon(v-u) \, dx,$$

$$\int_\Omega \nabla v \nabla(u\theta_\epsilon(v-u)) \, dx \leq \int_\Omega g(v)u\theta_\epsilon(v-u) \, dx$$

In addition, we can write

$$\int_\Omega g(u)v\theta_\epsilon(v-u) \, dx - \int_\Omega g(v)u\theta_\epsilon(v-u) \, dx \leq$$

$$\leq \int_\Omega \nabla u \nabla (v\theta_\epsilon(v-u))\,dx - \int_\Omega \nabla v \nabla (u\theta_\epsilon(v-u))\,dx =$$

$$= \int_\Omega v\theta'_\epsilon(v-u)\nabla u \cdot (\nabla v - \nabla u)\,dx - \int_\Omega u\theta'_\epsilon(v-u)\nabla v \cdot (\nabla v - \nabla u)\,dx =$$

$$= -\int_\Omega v\theta'_\epsilon(v-u)(\nabla u - \nabla v)^2\,dx + \int_\Omega (v-u)\theta'_\epsilon(v-u)\nabla v \cdot (\nabla v - \nabla u)\,dx \leq$$

$$\leq \int_\Omega \nabla v \nabla (\gamma_\epsilon(v-w))\,dx \leq \int_\Omega g(v)\gamma_\epsilon(v-u)\,dx$$

where $\gamma_\epsilon(t) = \int_0^t s\theta'_\epsilon(s)ds$. Since $0 \leq \gamma_\epsilon(t) \leq \epsilon$ for all $t \in \mathbb{R}$ and $g \in L^1(\Omega)$, we verify that

$$\int_\Omega vu\left(\frac{g(u)}{u} - \frac{g(v)}{v}\right)\theta_\epsilon(v-u)\,dx \leq \epsilon.$$

Taking the limit $\epsilon \to 0$, we can write

$$\int_{[v>u]} vu\left(\frac{g(u)}{u} - \frac{g(v)}{v}\right)\,dx \leq 0,$$

where we have denoted the set $\{x \in \Omega\,; v(x) > u(x)\}$ by $[v > u]$. This, in turn, implies that the measure of the set $[v > u]$ is zero. Thus, $u \geq v$ a.e. in Ω and the proof is complete. $\square$

Now, we shall prove a result used directly in our application of the Galerkin Method. For that, we need first the famous Brouwer fixed point theorem (see, for example, [35, Theorem 5.2.3.]).

Theorem 2.0.4 (Brouwer)**.** *Let $f : \overline{B_r(x)} \longrightarrow \overline{B_r(x)}$ be a continuous function defined on $\overline{B_r(x)} \subset \mathbb{R}^m$. Then, there exists $z \in \overline{B_r(x)}$ such that $f(z) = z$, that is, z is a fixed point of f.*

With it, we can prove the following result, known as the Fundamental Lemma.

Lemma 2.0.1. *Let $h : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be a continuous function such that $\langle h(\alpha), \alpha\rangle \geq 0$ for every $\alpha \in \mathbb{R}^m$ with $|\alpha| = R$, for some $R > 0$. Then there exists an element $z \in \overline{B_R(0)}$ such that $h(z) = 0$.*

Proof. If we suppose $f(x) \neq 0$ in $\overline{B_r(x)}$, we can then consider the function $g : \overline{B_r(x)} \longrightarrow \mathbb{R}^N$ defined by

$$g(x) = -\frac{R}{|f(x)|}f(x).$$

It is well-defined and continuous in $B_R(0)$. Besides, we can see that

$$|g(x)| = \frac{R}{|f(x)|}|f(x)| = R.$$

By Brouwer Fixed Point Theorem, there exists $z \in B_R(0)$ such that $g(z) = z$, which implies that

$$R^2 \;=\; |g(z)|^2 \;=\; \langle g(z), g(z)\rangle \;=\; \langle z, g(z)\rangle \;=\; -\frac{R}{|f(z)|}\langle z, f(z)\rangle.$$

Since $\langle z, f(z)\rangle \geq 0$, we obtain

$$0 < R^2 \;=\; -\frac{R}{|f(z)|}\langle z, f(z)\rangle \;\leq\; 0$$

which is a contradiction. Thus, $x_0 \in \overline{B_r(x)}$ must be such that $f(x_0) = 0$. □

We are now in position to give the proofs of our two main theorems and we do this in the following sections.

2.1 PROOF OF THEOREM 2.0.1

Our goal is to use the Galerkin Method to prove Theorem 2.0.1. For that, we will need to define, with the help of the real sequence defined in (H_4), a sequence of auxiliary equations that will be important for our purpose. More specifically, for each $k \in \mathbb{N}$, we define the auxiliary truncation functions by choosing $r \in (1, 2^* - 1)$ such that $2^* - 1 - r < \theta$ and we set

$$f_k(t) = \begin{cases} 0, & t \leq 0, \\ f(t), & 0 \leq t \leq M_k \\ \frac{f(M_k)}{(M_k)^r} t^r, & t \geq M_k. \end{cases} \tag{2.3}$$

Remark 2.1.1. Notice that we define f_k to be such that r in its definition is independent of k. We see that we are really truncating our original function, making it subcritical for large arguments. Furthermore, in view of conditions (H_3), (H_4) and the choice of r, we can prove that, for k big enough, f_k satisfies, for a constant $C > 0$,

$$|f_k(v)| \leq C(M_k)^{2\theta}|v|^r. \tag{2.4}$$

Indeed, for all $t > 0$, condition (H_4) and (2.3) gives

$$f_k(t) \leq \frac{f(M_k)}{(M_k)^r} t^r$$

and, by (H_3), if k is sufficiently large,

$$\frac{f(M_k)}{(M_k)^r} \leq C(M_k)^{2^*-1-r+\theta} \leq C(M_k)^{2\theta}.$$

For each $k \in \mathbb{N}$, let us consider the following auxiliary problem

$$\begin{cases} -\Delta u = \lambda u^{q-1} + f_k(u) + \sigma\omega, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ B(u) = 0, & x \in \partial\Omega, \end{cases} \tag{$P_{k,\sigma}$}$$

where $\sigma > 0$ is a real parameter and $\omega \in C_0^\infty(\Omega)$ is a positive fixed function.

Remark 2.1.2. Let us see that if u is a solution to $(P_{k,0})$ and is such that $|u|_{+\infty} \leq M_k$, then we actually obtain that u is a solution to our main problem (1.11). This will be exactly our approach at the end of this chapter when we seek to recover regularity of our solution.

To carry out the process of finding the solution to (P_1), we must first look for a solution to each equation $(P_{k,\sigma})$. For that, let $\beta = \{e_1, e_2, \ldots, e_n, \ldots\}$ be an orthonormal basis of $E_{\Sigma_1}(\Omega)$ and we define the subspace $V_m = [e_1, e_2, \ldots, e_m]$ of $E_{\Sigma_1}(\Omega)$ as being generated by the first m vectors of β and equipped with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2\, dx\right)^{\frac{1}{2}}$ already mentioned. With these conditions, each V_m, being a finite Hilbert Space, is isomorphic to $\mathbb{R}^m$. That allows us to define the function $F : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ whose coordinate functions are

$$F_j(\alpha) = \int_\Omega \nabla u \nabla e_j\, dx - \lambda \int_\Omega (u_+)^{q-1} e_j\, dx - \int_\Omega f_k(u_+) e_j\, dx - \sigma \int_\Omega \omega e_j\, dx, \quad j = 1, 2, \ldots, m, \tag{2.5}$$

where $u = \sum_{i=1}^m \alpha_i e_i$ is the function in V_m related to $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ through the isomorphism mentioned above.

To get the desired results, we shall also apply Lemma 2.0.1 to the function F, so the first step here is to show that it satisfies its conditions. Let us first check continuity.

Proposition 2.1.1. *The function $F : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ defined in* (2.5) *is continuous.*

Proof. Since we have already mentioned the isomorphism between $\mathbb{R}^m$ and V_m, we can consider a sequence converging $v_n \to v$ in V_m and it is sufficient to prove that each $F_j(v_n)$ converges to $F_j(v)$. Notice that this is immediate for the first, second, and last term of F_j, using the Holder Inequality and the Sobolev Embedding Theorems. What we have to show is only the continuity of the term $\int_\Omega f_k(u_+) e_j\, dx$, but here we can use the fact that each f_k satisfies condition (2.4). By that, for any sequence v_n, we shall have

$$|f_k(v_{n+}) e_j| \leq C_k |v_{n+}|^r |e_j| \leq C_{1,k} |v_n|^{2^*} + C_{2,k} |e_j|^{\frac{2^*}{2^*-r}},$$

where was used Young's inequality with conjugate exponents $\frac{2^*}{r}$ and $\frac{2^*}{2^*-r}$. Thus, if we have a sequence $v_n \to v$ in some V_m, we know that $\|v_n\|$, and therefore $|v_n|_{2^*}$ is bounded, so that

$$|f_k(v_{n+}) e_j| \leq g_k\ , \qquad \text{for some } g_k \in L^1(\Omega).$$

From the fact that this bound does not depend on n, the DCT readily implies that

$$\int_\Omega f_k(v_{n+}) e_j\, dx \xrightarrow{n \to +\infty} \int_\Omega f_k(v+) e_j\, dx, \tag{2.6}$$

proving the continuity of F. □

Now, what remains to be done is prove the following proposition.

Proposition 2.1.2. *There exists a real number $R > 0$ such that, for $|\alpha| = R$, we have $\langle F(\alpha), \alpha\rangle \geq 0$.*

Proof. By definition, we have

$$\begin{aligned}
\langle F(\alpha), \alpha\rangle &= \sum_{i=1}^{m} F_i(\alpha)\alpha_i \\
&= \sum_{i=1}^{m} \left(\int_\Omega \nabla u \nabla e_i \, dx - \lambda \int_\Omega (u_+)^{q-1} e_i \, dx \right. \\
&\quad \left. - \int_\Omega f_k(u_+) e_i \, dx - \sigma \int_\Omega \omega e_i \, dx \right) \alpha_i \\
&= \int_\Omega \nabla u \nabla \left(\sum_{i=1}^{m} e_i \alpha_i \right) dx - \lambda \int_\Omega (u_+)^{q-1} \left(\sum_{i=1}^{m} e_i\alpha_i \right) dx \\
&\quad - \int_\Omega f_k(u_+) \left(\sum_{i=1}^{m} e_i\alpha_i \right) dx - \sigma \int_\Omega \omega \left(\sum_{i=1}^{m} e_i\alpha_i \right) dx \\
&= \int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega (u_+)^q \, dx - \int_\Omega f_k(u_+) u_+ \, dx - \sigma \int_\Omega \omega u \, dx.
\end{aligned}$$

Again by the Sobolev Embedding Theorems, we can write

$$\int_\Omega u_+^q \, dx \;\leq\; \int_\Omega |u|^q \, dx \;\leq\; C_1 \|u\|^q \qquad \text{and} \qquad \int_\Omega \omega u \, dx \;\leq\; C_2 \|u\| \tag{2.7}$$

so that

$$\langle F(\alpha), \alpha\rangle \;\geq\; \|u\|^2 - \lambda C_1 \|u\|^q - \int_\Omega f_k(u) u \, dx - \sigma C_2 \|u\|,$$

where we remember that $1 < q < 2 < 2^*$. By relation (2.4), we can also estimate the term with the auxiliary functions by

$$\int_\Omega f_k(u) u \, dx \leq C(M_k)^{2\theta} \int_\Omega |u|^{r+1} \, dx \leq C_3 \|u\|^{r+1}. \tag{2.8}$$

Now, observing the behavior of the function $g(t) = t^2 - \lambda C_1 t^q - C_3 t^{r+1}$, it is evident, since $r + 1 > 2$, that $h(t) = t^2 - C_3 t^{r+1} > 0$ for every $0 < t < C_3^{\frac{1}{1-r}}$. Besides that, a quick study of its derivatives shows us that

$$R = \left(\frac{2}{C_3\,(r+1)} \right)^{\frac{1}{r-1}} \;<\; C_3^{\frac{1}{1-r}}$$

is a local maximum and $h(R) > 0$. Thus, considering $\lambda > 0$ and $\sigma^* > 0$ such that

$$0 < \lambda \;<\; \Lambda = \frac{R^{2-q} - C_3 R^{r+1-q}}{C_1}, \qquad \sigma C_2 R < R^2 - \lambda C_1 R^q - C_3 R^{r+1},$$

we have

$$\langle F(\alpha), \alpha\rangle \geq \; \|u\|^2 - \lambda C_1 \|u\|^q - C_3 \|u\|^{r+1} - \sigma C_2 \|u\| \;>\; 0$$

provided $|\alpha| = \|u\| = R$, as intended. $\square$

Consequently, we have found a sequence of functions $v_m \in V_m$, with $\|v_m\| \leq R$, such that $F(v_m) = 0$, i.e., for each $j = 1, 2, ..., m$,

$$\int_\Omega \nabla v_m \nabla e_j \, dx - \lambda \int_\Omega (v_{m+})^{q-1} e_j \, dx - \int_\Omega f_k(v_{m+}) e_j \, dx - \sigma \int_\Omega \omega e_j \, dx = 0. \tag{2.9}$$

It is easy to see that, from the linearity of the expressions in the left side of (2.9), we can expand the result to all V_m, that means

$$\int_\Omega \nabla v_m \nabla \phi \, dx - \lambda \int_\Omega (v_{m+})^{q-1} \phi \, dx - \int_\Omega f_k(v_{m+}) \phi \, dx - \sigma \int_\Omega \omega \phi \, dx = 0, \quad \forall \, \phi \in V_m. \tag{2.10}$$

What we now have is a sequence $(v_m)_{m \in \mathbb{N}} \in E_{\Sigma_1}(\Omega)$ whose norm is bounded by the constant R. Since $E_{\Sigma_1}(\Omega)$ is a Hilbert Space, it is also reflexive and therefore it is weakly compact. This means that we can obtain a subsequence, which we will still denote by (v_m), and a function $v \in E_{\Sigma_1}(\Omega)$, such that

$$v_m \rightharpoonup v \text{ in } E_{\Sigma_1}(\Omega) \qquad \text{and} \qquad v_m \to v \text{ in } L^s(\Omega), \quad s \in [1, 2^*], \tag{2.11}$$

where the second convergence was obtained by the Sobolev Embedding Theorems.

Now, using (2.4) and the same reasoning we applied to achieve (2.6), we have, passing the limit $m \to \infty$,

$$\lambda \int_\Omega (v_{m+})^{q-1} \phi_l \, dx \longrightarrow \lambda \int_\Omega (v_+)^{q-1} \phi_l \, dx,$$

$$\int_\Omega f_k(v_{m+}) \phi_l \, dx \longrightarrow \int_\Omega f_k(v_+) \phi_l \, dx.$$

Thus, we arrive at

$$\int_\Omega \nabla v \nabla \phi_l \, dx - \lambda \int_\Omega (v_+)^{q-1} \phi_l \, dx - \int_\Omega f_k(v_+) \phi_l \, dx - \sigma \int_\Omega \omega \phi_l \, dx = 0, \quad \forall \, \phi \in V_l.$$

We can notice that this last equation is true for every $l \in \mathbb{N}$. By the density of the spaces V_l in $E_{\Sigma_1}(\Omega)$, we can relate to each $\phi \in E_{\Sigma_1}(\Omega)$ a sequence $(\phi_l)_{l \in \mathbb{N}}$ with each $\phi_l \in V_l$ and passing the limit $l \to \infty$, we achieve

$$\int_\Omega \nabla v \nabla \phi \, dx - \lambda \int_\Omega (v_+)^{q-1} \phi \, dx - \int_\Omega f_k(v_+) \phi \, dx - \sigma \int_\Omega \omega \phi \, dx = 0, \quad \forall \, \phi \in E_{\Sigma_1}(\Omega).$$

At last, we can show that $v(x) \geq 0$ for every $x \in \Omega$. This is evident, since, using $v_-(x) = \max\{0, -v(x)\}$ as a test function,

$$-\|v_-\|^2 = \int_\Omega \nabla v \nabla (v_-) \, dx = \int_\Omega (v_+)^{q-1} v_- \, dx + \int_\Omega f_k(v_+) v_- \, dx + \sigma \int_\Omega \omega v_- \, dx \;\geq\; 0,$$

showing that $v = v_+$. In particular, this implies that we must have, apart from a null set, $v \geq 0$ in the boundary $\partial\Omega$. Furthermore, to show that v is strictly positive we can use the

Maximum Principles applied to the set Ω. If we suppose there exists a point $x \in \Omega$ for which $v(x) = 0$, then we would conclude that v must be a constant function. But, this would imply $\sigma\omega = 0$, which is a contradiction. By that, this function must be strictly positive in Ω, thus being a weak solution to the problem $(P_{k,\sigma})$. To outline the dependence of the parameter σ, we denote this function by v_σ.

Since each v_m obtained in (2.9) must be bounded in the $E_{\Sigma_1}(\Omega)$ norm by a constant, so will be each v_σ, and since this bound is uniform (that is, does not depend on σ) we can do the same reasoning as in (2.11) to the sequence of functions v_σ to obtain a function in $E_{\Sigma_1}(\Omega)$, which is the weak solution of the problem

$$\begin{cases} -\Delta u = \lambda u^{q-1} + f_k(u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ B(u) = 0, & x \in \partial\Omega, \end{cases} \tag{P_k}$$

and which we will denote by v_k, again to reassure its dependence of the index k in the auxiliary equation.

It is also evident that, taking the limit $\sigma \to 0$, we get $v_k \geq 0$ in Ω. More than that, each solution of $(P_{k,\sigma})$ is evidently a weak supersolution of the problem

$$\begin{cases} -\Delta u = \lambda u^{q-1}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ B(u) = 0, & x \in \partial\Omega, \end{cases} \tag{2.12}$$

which we know to have a classical solution $w \in E_{\Sigma_1}(\Omega)$ and, therefore, a weak subsolution. By Theorem 2.0.3, we can assure that $v_\sigma \geq w > 0$, which in turn leads to $v_k \geq w > 0$ in Ω. It is important to observe that this result does not depend on the index k.

Now, we can proceed with the proof of Theorem 2.0.1, extending the results to the main problem (P_1). First we notice that, for each $k \in \mathbb{N}$, taking $f_1 = \lambda v^{q-1} + f_k(v)$, we have

$$|f_1| = |\lambda v^{q-1} + f_k(v)| \leq \lambda |v|^{q-1} + C(M_k)^{2\theta}|v|^r.$$

Thus, the nonlinearity of each problem (P_k) is bounded in $L^{\frac{2^*}{r}}(\Omega)$. With that, we conclude that $v_k \in W^{2,\frac{2^*}{r}}(\Omega)$ and

$$\|v_k\|_{W^{2,\frac{2^*}{r}}(\Omega)} \leq D_1 \left(|f_1|_{L^{\frac{2^*}{r}}(\Omega)} + |v_k|_{L^{\frac{2^*}{r}}(\Omega)} \right).$$

Now, since

$$\|v_k\| \ \leq \ \liminf_{m\to\infty} \|v_m\| \ \leq \ R, \tag{2.13}$$

we can use the Sobolev Embedding Theorems to prove that

$$\|v_k\|_{W^{2,\frac{2^*}{r}}(\Omega)} \leq D_2 M_k^{2\theta}.$$

What we do next is prove that, for a large k, $\limsup_{x\in\Omega}|v_k(x)| \leq M_k$, so that v_k can actually be considered a solution of the main problem (P_1). We do this by the method of Bootstrapping, which proceeds as follows: if $\frac{2^*}{r} > \frac{N}{2}$, then we have $W^{2,\frac{2^*}{r}}(\Omega) \hookrightarrow C^\gamma(\overline{\Omega})$ and since Ω is bounded,

$$\|v_k\|_{L^\infty} \leq D_2 M_k^{2\theta},$$

so that, if $\theta \in (0, \frac{1}{2})$, we can write, provided that M_k is large enough,

$$\|v_k\|_{L^\infty} \leq M_k.$$

If $\frac{2^*}{r} = \frac{N}{2}$, then $W^{2,\frac{2^*}{r}}(\Omega) \hookrightarrow L^t(\Omega)$ for every $t \in [1,\infty)$. But, this implies that we can take $t > \frac{N}{2}r$ such that $\lambda v^{q-1} + f_k(v) \in L^{\frac{t}{r}}(\Omega)$ and consequently $v_k \in W^{2,\frac{t}{r}}(\Omega)$. By the same argument as above,

$$\|v_k\|_{W^{2,\frac{t}{r}}(\Omega)} \leq D_3 \left(|f|_{L^{\frac{t}{r}}(\Omega)} + |v_k|_{L^{\frac{t}{r}}(\Omega)}\right)$$

and by the estimates on f_k,

$$\|v_k\|_{W^{2,\frac{t}{r}}(\Omega)} \leq D_3 \left(M_k^{(2\theta)(r+1)} + M_k^{2\theta}\right).$$

Taking $\theta \in (0, \frac{1}{2(1+r)})$, we get for M_k large enough

$$\|v_k\|_{L^\infty} \leq \|v_k\|_{W^{2,\frac{2^*}{r}}(\Omega)} \leq M_k.$$

At last, for the case $p = \frac{2^*}{r} < \frac{N}{2}$, we will apply the former cases in a iterative process. We note first that since $r \in \left(1, \frac{N+2}{N-2}\right)$, there must existe a $\epsilon > 0$ such that

$$p = (1+\epsilon)\frac{2N}{N+2}.$$

By the Sobolev-Morrey Embeddings, we have

$$W^{2,p}(\Omega) \hookrightarrow L^{s_1}(\Omega)\ , \quad s_1 = \frac{Np}{N-2p},$$

implying that $v_k \in L^{s_1}(\Omega)$ and, thus, $\lambda(v_k)^{q-1} + f_k(v_k) \in L^{\frac{s_1}{r}}(\Omega)$. Consequently, $v_k \in W^{2,p_1}(\Omega)$, where we defined $p_1 = \frac{s_1}{r}$.

To see that we have elevated the regularity of v_k, we notice that

$$\frac{p_1}{p} = \frac{s_1}{2^*} = \left(\frac{Np}{N-2p}\right)\left(\frac{N-2}{2N}\right) = \frac{(1+\epsilon)(N-2)}{N-2-4\epsilon} > 1+\epsilon.$$

We can again expect p_1 to fall into one of those three cases regarding its relation with $\frac{N}{2}$. The first two cases will give us the desired result, just as before. If, however, we have again $p_1 < \frac{N}{2}$, we now reason that, by the same arguments as before,

$$v_k \in W^{2,p_2}(\Omega)\ , \qquad \text{where } p_2 = \frac{s_2}{r}\ , \quad s_2 = \frac{Np_1}{N-2p_1}.$$

Again, we see that

$$\frac{p_2}{p_1} = \frac{Np_1(N-2p)}{Np(N-2p_1)} > (1+\epsilon)\left(\frac{N-2p}{N-2p_1}\right) > 1+\epsilon.$$

We can show that, repeating this argument a finite number of times, we prove that $v_k \in W^{2,p'}(\Omega)$ for some $p' \geq \frac{N}{2}$ and we will obtain one of the two first cases, proving that there exist a number γ such that $|v_k|_\infty \leq M_k$ for some large k, provided $\theta \in (0,\gamma)$.

We have then completed the proof of Theorem 2.0.1.

2.2 PROOF OF THEOREM 2.0.2

Assume, by contradiction, that $\lambda^* = +\infty$. This means that there exists a sequence $\lambda_n \to +\infty$ and solutions $u_n \in E_{\Sigma_1}(\Omega) \cap W^{2,\frac{2^*}{2^*-1}}(\Omega)$ to problem (P_1), with $u_n > 0$ in Ω for each n.

Fix $0 < \xi < 2^* - 2$ and $\lambda_0 > 1$. Define the auxiliary function

$$P_\lambda(t) = \frac{\lambda}{\lambda_0} t^{q-1} + t^{1+\xi},\ t > 0.$$

Notice that, for λ big enough,

$$\lambda t^{q-1} + f(t) \geq P_\lambda(t), \quad \text{for} \quad t > 0. \tag{2.14}$$

Indeed, we begin by noticing that, by condition (H_2), there exists $t_0 > 1$ such that

$$f(t) \geq t^{2^*-1}, \quad \text{for} \quad t > t_0.$$

With this, we can divide our interval $(0,+\infty)$ into three parts.

$t \in (0,1)$: For $t \in (0,1)$, we have $t^{q-1} > t^{1+\xi}$, so that if λ is such that $\lambda(1-\lambda_0^{-1}) > 1$, we shall have

$$\lambda\left(1 - \frac{1}{\lambda_0}\right) t^{q-1} + f(t) \geq t^{q-1} > t^{1+\xi}.$$

$t \in (t_0,+\infty)$: Since $t_0 > 1$, for $t \in (t_0,+\infty)$, we have just seen that $f(t) \geq t^{2^*-1} > t^{1+\xi}$ and therefore we have again the desired inequality.

$t \in [1,t_0]$: At last, for $t \in [1,t_0]$, we can take $\lambda > \frac{\lambda_0 t_0^{1+\xi}}{\lambda_0 - 1}$, meaning we obtain

$$\lambda\left(1 - \frac{1}{\lambda_0}\right) t^{q-1} + f(t) \geq \lambda\left(1 - \frac{1}{\lambda_0}\right) \geq t_0^{1+\xi} \geq t^{1+\xi}.$$

With this, we have just shown that choosing $\lambda > \max\{\frac{\lambda_0}{\lambda_0-1}, \frac{\lambda_0 t_0^{1+\xi}}{\lambda_0-1}\}$, the inequality in (2.14) is valid for all values of $t > 0$.

Now, let us see that there exists a constant $C_\lambda > 0$ such that

$$\lambda P_\lambda(t) \geq C_\lambda t, \quad \text{for} \quad t > 0. \tag{2.15}$$

Indeed, let us consider the function

$$Q_\lambda(t) = P_\lambda(t)t^{-1} = \frac{\lambda}{\lambda_0} t^{q-2} + t^\xi.$$

It is evident that $Q_\lambda(t) \to \infty$ as $t \to 0^+$, as well as in the limit $t \to \infty$. Furthermore, let t_1 be such that $Q_\lambda(t_1) = C_\lambda$ is the minimum value of Q_λ, meaning $t_1 > 0$ is the unique root of

$$\frac{\lambda}{\lambda_0}(q-2)t^{q-3} + \xi t^{\xi-1} = 0.$$

This gives us

$$t_1 = \left(\frac{\lambda(2-q)}{\lambda_0 \xi}\right)^{\frac{1}{2+\xi-q}}, \qquad C_\lambda = \lambda^{\frac{\xi}{2+\xi-q}} \left[\frac{1}{\lambda_0}\left(\frac{2-q}{\lambda_0\xi}\right)^{\frac{q-2}{2+\xi-q}} + \left(\frac{2-q}{\lambda_0\xi}\right)^{\frac{\xi}{2+\xi-q}}\right].$$

Notice that t_1 increases as λ increases, since $q < 2$, and the constant C_λ has the same behavior with respect to λ. Here we are considering λ sufficiently large.

Let $\sigma_1 > 0$ be the first eigenvalue of the Laplacian and $\varphi_1 > 0$ the associated first eigenfunction satisfying

$$\begin{cases} -\Delta\varphi_1 = \sigma_1\varphi_1 & \text{in} \quad \Omega \\ B(\varphi_1) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Since $C_{\lambda_n} \to \infty$ as $\lambda_n \to \infty$, for each given $\delta > 0$, there is λ_{n_0} such that $C_{\lambda_{n_0}} \geq \sigma_1 + \delta$. Hence the solution $u_{n_0} > 0$ of (P_1) corresponding to λ_{n_0} satisfies

$$\begin{cases} -\Delta u_{n_0} \geq C_{\lambda_{n_0}} u_{n_0} \geq (\sigma_1 + \delta)u_{n_0} & \text{in} \quad \Omega \\ B(u_{n_0}) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

On the other hand, taking $\varepsilon \in (0,1)$ small enough we obtain $\varepsilon\varphi_1 < u_{n_0}$ in Ω, this being possible because $u_{n_0} \geq \varphi_1$ and $\partial\varphi_1/\partial\nu < 0$ on $\partial\Omega$. Furthermore, we have

$$\begin{cases} -\Delta(\varepsilon\varphi_1) = (\varepsilon\sigma_1)\varphi_1 \leq (\sigma_1 + \delta)(\varepsilon\varphi_1) & \text{in} \quad \Omega, \\ B(\varphi_1) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and hence $\varepsilon\varphi_1$ is a sub-solution. By the sub-supersolution method (for a more detailed discussion of this method for different kinds of boundary condition, see [36]), there is a solution $\varepsilon\varphi_1 < \zeta < u_{n_0}$ in Ω of

$$\begin{cases} -\Delta\zeta = (\sigma_1 + \delta)\zeta & \text{in} \quad \Omega \\ B(\zeta) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We thus have a contradiction to the fact that σ_1 is isolated (the fact that σ_1 is isolated for the Dirichlet problem is very well know, but we can obtain the same results for the Neumann boundary conditions and, most importantly, for the mixed boundary conditions problem. For that, see, for instance, [16] and the references it cites).

We conclude from this that we must have $\lambda^* < \infty$.

3 SOLUTION FOR A GENERALIZED SCHRÖDINGER-POISSON SYSTEM INVOLVING BOTH SINGULAR AND GENERAL NONLINEARITIES

Now, we look at our second problem, discussing the existence of positive solutions to the following Schrödinger-Poisson system,

$$\begin{cases} -\Delta u - \phi u^{2^*-2} = \frac{\lambda}{u^\gamma} & \text{in} \quad \Omega, \\ -\Delta \phi = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{P_2}$$

where $\lambda > 0$ and $\gamma \in (0,1)$ are real parameters. As we said, we will consider f to be continuous and satisfying the critical growth condition

$$0 \leq f(s)s \leq L|s|^{2^*}, \qquad L > 0. \tag{3.1}$$

The presence of a singularity brings obvious complications as to the possibility of a blow up in certain points in Ω. In addition, there is also the difficulty of treating critical growth, since such terms cause the lack of compactness of the spaces dealt with and thus hinder, as we have seen in Chapter 1, the use of variational methods. The use of the non-variational Galerkin Method helps again avoid this problem with the critical growth and at the same time brings no insurmountable difficulties to dealing with the singular term. We do this while still keeping the nonlinearity f in the second equation of (P_2) quite general and not asking additional conditions like, for example, the Ambrosetti-Rabinowitz growth condition, commonly adopted in elliptic problems.

We restate our main result, already introduced in Chapter 1.

Theorem 3.0.1. *If $f : [0,\infty) \longrightarrow \mathbb{R}$ is a continuous function satisfying the growth condition* (3.1)*. Then there exists $\Lambda > 0$ such that, for every $0 < \lambda < \Lambda$, problem (P_2) has a pair of solutions $u_\lambda, \phi_\lambda \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$, with $p = \frac{2^*}{2^*-1}$.*

We notice the striking similarity between the conditions for which our result and Theorems 1.4.4 hold. It is, therefore, safe to say that our findings pose as a generalization of theirs.

3.1 PRELIMINARY RESULTS AND AUXILIARY SOLUTIONS

Firstly, we present the comparison result due to Ambrosetti, Brezis and Cerami (for the proof, we refer the reader to [34, Lemma 3.3.]), which will play an important role in the proof of our main theorem.

Lemma 3.1.1 (Ambrosetti, Brézis and Cerami). *Consider $g\colon \mathbb{R} \to \mathbb{R}$ a continuous function satisfying $t^{-1}g(t)$ decreasing for $t > 0$. If $u_1, u_2 \in C^2(\Omega)$ are strong sub and supersolution, respectively, of the problem below*

$$\begin{cases} -\Delta u = g(u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.2}$$

meaning that we have

$$\begin{cases} -\Delta u_1 \le g(u_1), & x \in \Omega, \\ u_1 > 0, & x \in \Omega, \\ u_1 = 0, & x \in \partial\Omega, \end{cases} \tag{3.3}$$

$$\begin{cases} -\Delta u_2 \ge g(u_2), & x \in \Omega, \\ u_2 > 0, & x \in \Omega, \\ u_2 = 0, & x \in \partial\Omega. \end{cases} \tag{3.4}$$

Then $u_2 \ \geq \ u_1,\ x \in \Omega$.

In the final part of the present chapter, we shall also need the following result, which we shall only enunciate here.

Theorem 3.1.1. *Suppose that $p \in (1, +\infty)$ and that $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions in $L^p(\Omega)$ such that $(|f_n|_p)_{n\in\mathbb{N}}$ is a bounded sequence of numbers. If $f_n \to f$ a.e. in Ω, then $f_n \rightharpoonup f$ in $L^p(\Omega)$.*

Proof. See [37, Theorem 13.44]. □

Let us now turn our attention back to our main problem. Evidently, the singularity present in the first equation of (P_2) elevates the complexity of the problem in question. It is what motivates the condition $u(x) > 0$ for every $x \in \Omega$ and is one of the principal reasons we must first solve a sequence of auxiliary equations. For each $k \in \mathbb{N}$, we shall consider first the following

$$\begin{cases} -\Delta u - \phi u^{2^*-2} = \frac{\lambda}{(u+\frac{1}{k})^\gamma}, & x \in \Omega, \\ -\Delta \phi = f_k(u), & x \in \Omega, \\ u > 0, & x \in \partial\Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases} \tag{P_k}$$

where f_k is a sequence of auxiliary functions, given by

$$f_k(s) = \begin{cases} -k[G(-k-\frac{1}{k}) - G(-k)], & if \quad s \leq -k \\ -k[G(s-\frac{1}{k}) - G(s)], & if \quad -k \leq s \leq -\frac{1}{k} \\ k^2 s[G(-\frac{2}{k}) - G(-\frac{1}{k})], & if \quad -\frac{1}{k} \leq s \leq 0 \\ k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})], & if \quad 0 \leq s \leq \frac{1}{k} \\ k[G(s+\frac{1}{k}) - G(s)], & if \quad \frac{1}{k} \leq s \leq k \\ k[G(k+\frac{1}{k}) - G(k)], & if \quad s \geq k, \end{cases} \tag{3.5}$$

with $G(s) = \int_0^s f(\xi)d\xi$. Here, not only the singularity, but also the allowed growth condition for f, make it necessary to consider f_k instead of f, the former having a much higher regularity. More specifically, this sequence, for which we reference the reader to [12] for a good example of its application, has the following properties. The proof of the first lemma can be seen in [28].

Lemma 3.1.2. *The sequence of auxiliary functions $f_k : \mathbb{R} \to \mathbb{R}$ given above is such that*

1. $sf_k(s) \geq 0$ *for* $s \in \mathbb{R}, \quad k \in \mathbb{N}$.
2. *For all* $k \in \mathbb{N}$*, there exists* $c_k \in \mathbb{R}$ *such that* $|f_k(t) - f_k(s)| \leq c_k|t-s|$*, for* $s, t \in \mathbb{R}$.
3. $f_k \longrightarrow f$ *uniformly in any bounded subset of* $\mathbb{R}$.

This result will prove to be crucial when we study the regularity of the solutions we obtain. In addition, we can also state and demonstrate the following lemma about the estimates of the sequence f_k.

Lemma 3.1.3. *The sequence of auxiliary functions f_k defined above satisfies*

1. $\forall\, k \in \mathbb{N},\ 0 \ \leq\ s\ f_k(s) \ \leq\ L_1|s|^{2^*}, \quad |s| \geq \frac{1}{k}$,
2. $\forall\, k \in \mathbb{N},\ 0 \ \leq\ s\ f_k(s) \ \leq\ L_2|s|^2, \quad |s| \leq \frac{1}{k}$,

where L_1 and L_2 are positive constants independent of k.

Proof. To prove this result, we must divide our considerations into different cases.

First Case: Consider $-k \leq s \leq -\frac{1}{k}$.

Using the mean value theorem, there exists $\xi \in (s - \frac{1}{k}, s)$ such that

$$f_k(s) = k(G(s) - G(s - \frac{1}{k}) = -k\frac{d}{ds}G(\xi)\frac{1}{k} = f(\xi),$$

which proves that $sf_k(s) = sf(\xi)$ in this interval. Furthermore, since $\xi < s$ and $f(\xi) < 0$, we have

$$\begin{aligned} sf_k(s) \leq sf(\xi) \leq \xi f(\xi) &\leq L\,|\xi|^{2^*} \\ &\leq L\left|s - \frac{1}{k}\right|^{2^*} \\ &\leq L(|s| + \frac{1}{k})^{2^*} \\ &\leq L(2\,|s|)^{2^*} \\ &\leq 2^{2^*} L\,|s|^{2^*}. \end{aligned} \tag{3.6}$$

Second Case: Consider $\frac{1}{k} \leq s \leq k$.

As before, there must exist $\xi \in (s, s + \frac{1}{k})$ such that

$$f_k(s) = k(G(s + \frac{1}{k}) - G(s) = k\frac{d}{ds}G(\xi)\frac{1}{k} = f(\xi),$$

proving that, again, $sf_k(s) = sf(\xi)$ in this interval. Now, given that $s < \xi$ and $f(\xi) > 0$, we have

$$\begin{aligned} sf_k(s) \leq sf(\xi) \leq \xi f(\xi) &\leq L\,|\xi|^{2^*} \\ &\leq L\left|s + \frac{1}{k}\right|^{2^*} \\ &\leq 2^{2^*} L\,|s|^{2^*}. \end{aligned} \tag{3.7}$$

Third Case: Consider $s \geq k$.

In this case, we choose $\xi \in (k, k + \frac{1}{k})$ such that

$$f_k(s) = k(G(k + \frac{1}{k}) - G(k) = k\frac{d}{ds}G(\xi)\frac{1}{k} = f(\xi),$$

obtaining, again, $sf_k(s) = sf(\xi)$. Now, we can write

$$sf_k(s) = \frac{s}{\xi}\xi f(\xi) \leq \frac{|s|}{|\xi|}L\,|\xi|^{2^*} \leq L\,|s|\,|\xi|^{2^*-1}.$$

Since $\xi < k + \frac{1}{k} \leq s + \frac{1}{k}$, we obtain

$$\begin{aligned} sf_k(s) \leq L\,|s|\left|s + \frac{1}{k}\right|^{2^*-1} &\leq L\,|s|\,2^{2^*-1}\,|s|^{2^*-1} \\ &\leq 2^{2^*-1}L\,|s|^{2^*}. \end{aligned} \tag{3.8}$$

Fourth Case: Consider $s \leq -k$.

This case is quite similar to the third one. Taking now $\xi \in (k - \frac{1}{k}, -k)$ such that

$$f_k(s) = k(G(k + \frac{1}{k}) - G(k) = k\frac{d}{ds}G(\xi)\frac{1}{k} = f(\xi),$$

we have one more time that $sf_k(s) = sf(\xi)$. By the same reasoning as before,

$$sf_k(s) \leq \frac{|s|}{|\xi|} L |\xi|^{2^*} \leq L |s| |\xi|^{2^*-1}.$$

Since $|\xi| < k + \frac{1}{k} \leq s + \frac{1}{k}$,

$$sf_k(s) \leq L |s| \left| s + \frac{1}{k} \right|^{2^*-1} \leq L |s| 2^{2^*-1} |s|^{2^*-1} \leq 2^{2^*-1} L |s|^{2^*}. \tag{3.9}$$

With this, we finally conclude item (i), taking L_1 to be $2^{2^*}L$. For item (ii), let us look at the last case.

Fifth Case: Consider $|s| \leq \frac{1}{k}$.

With the purpose of not dividing this proof into two more, very similar, cases, let us consider $s \geq 0$, leaving to the reader the evident generalization. There, we have

$$f_k(s) = k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})]$$

and, again by the mean value theorem, there exists $\xi \in (\frac{2}{k}, \frac{1}{k})$ such that

$$f_k(s) = k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})] = k^2 s \frac{d}{ds} G(\xi) \frac{1}{k} = ksf(\xi).$$

Now, we obtain

$$sf_k(s) = ks^2 f(\xi) \leq k \frac{|s|^2}{|\xi|} L |\xi|^{2^*} \leq kL |s|^2 |\xi|^{2^*-1}.$$

By the conditions on ξ,

$$sf_k(s) \leq kL |s|^2 \left| \frac{2}{k} \right|^{2^*-1} = k^{-2^*} L 2^{2^*-1} |s|^2 \leq L 2^{2^*-1} |s|^2. \tag{3.10}$$

By this, taking L_2 to be $2^{2^*-1}L$, we prove what was desired. □

What we intend to do, eventually, is to prove the existence of the sequence (u_k, ϕ_k), solutions to each (P_k), and subsequently show that we can obtain a pair $(u_\lambda, \phi_\lambda)$, the limit of a subsequence of (u_k, ϕ_k), which satisfies the condition for being solutions of the main problem (P_2). We remind that this last fact is characterized by the equalities

$$\int_\Omega \nabla u \nabla \omega \, dx - \int_\Omega \phi u^{2^*-2} \omega \, dx - \lambda \int_\Omega \frac{\omega}{u^\gamma} \, dx = 0,$$
$$\int_\Omega \nabla \phi \nabla \omega \, dx - \int_\Omega f(u) \omega \, dx = 0, \quad \omega \in H_0^1(\Omega).$$

Remark 3.1.1. The regularity of each pair (u_k, ϕ_k) is then an important factor to the final result, as is their sign in Ω. This is also the reason we must consider first the auxiliary functions f_k. Their regularity implies quite directly the strong regularity of each solution ϕ_k, which in turn does the same for u_k, as we shall see ahead. Furthermore, it is also known that even if each u_k is strictly positive, the same might not be true for its limit u. We will be able, nonetheless, to obtain the existence of an uniform lower bound for u_k, thus being able to achieve this goal.

Therefore, we shall present in this section the regularity of each pair (u_k, ϕ_k) of weak solutions to the auxiliary problem (P_k). The existence of the limit $(u_\lambda, \phi_\lambda)$ will be already assumed and will be proven in further sections.

First, we obtain an estimate to the nonlinearity of the first auxiliary equation (taking ϕ as a known function of x) by a subcritical growth in u, which will lead directly to the regularity of u_k. For each $k \in \mathbb{N}$, we define

$$H(u, \phi) = \phi u^{2^*-2} + \frac{\lambda}{(u + \frac{1}{k})^\gamma}$$

and $G(u, x) = H(u, \phi(x))$. The equation satisfied by u_k is then $-\Delta u = G(u, x)$. On the other hand, we have the following estimate

$$|H(u, \phi_k)| = \left| \phi_k u^{2^*-2} + \frac{\lambda}{(u + \frac{1}{k})^\gamma} \right| \leq |\phi_k||u|^{2^*-2} + \frac{\lambda}{|u + \frac{1}{k}|^\gamma} \leq |\phi_k||u|^{2^*-2} + \lambda k^\gamma.$$

This is the fundamental inequality we must have to be able to apply standard bootstrapping arguments and consequently show that $u_k, \phi_k \in C^2(\overline{\Omega})$ for every $k \in \mathbb{N}$. We do as follows:

We notice first that each f_k is a truncated function, being constant for $|s| \geq k$, meaning we have $f_k \in L^\infty(\mathbb{R})$, which in turn implies $f_k(u_k) \in L^\infty(\Omega)$ for every k (the fact that they are not uniformly bound does not hinder our development, since we are fixing k). By the second equation of (P_k) and standard results of regularity (see, for example, [38]), we obtain $\phi_k \in W^{2,r}(\Omega)$, for $r > 1$, and choosing r big enough we can obtain, through the Sobolev-Morrey Embeddings, $\phi_k \in C^{0,\alpha}(\overline{\Omega})$ for any $\alpha \in (0, 1]$. In particular, $\phi_k \in L^\infty(\overline{\Omega})$ and therefore[1]

$$|H(u, \phi_k)| \leq |\phi_k|_\infty |u|^{2^*-2} + \lambda k^\gamma. \tag{3.11}$$

We have seen that u_k satisfies weakly the equation

$$-\Delta u = G(u, x),$$

where, by inequality (3.11), we have $G(u_k(\cdot), \cdot) \in L^s(\Omega)$, with $s = \frac{2^*}{(2^*-2)}$. By known arguments of standard elliptic regularity, we obtain $u_k \in W^{2,s}(\Omega)$. We wish to show that we can elevate this regularity to $W^{2,s'}(\Omega)$ such that $2s' > N$.

If $2s > N$, we take $s' = s$ and there is nothing to be done.

If $2s \leq N$, we notice first that we can write

$$s = (1 + \epsilon)\frac{2^*}{2^* - 1} = (1 + \epsilon)\frac{2N}{N + 2}.$$

[1] Here we see the great advantage of dealing with f_k first. With them, the regularity of the auxiliary functions ϕ_k comes quite easily. Its boundness, caused by the boundness of each f_k, is also a crucial step in proving the regularity of u_k.

By the Sobolev-Morrey Embeddings, we can assert that $u_k \in L^{p_1}(\Omega)$, with

$$p_1 = \frac{Ns}{N-2s},$$

which in turn leads to $G(u_k(\cdot),\cdot) \in L^{s_1}(\Omega)$, $s_1 = \frac{p_1}{(2^*-2)}$. This shows that $u_k \in W^{2,s_1}(\Omega)$ and to see that we elevated the regularity, we write

$$\frac{s_1}{s} = \frac{p_1}{2^*} = \frac{Ns}{N-2s}\frac{N-2}{2N} = \frac{(1+\epsilon)(N-2)}{N-2-4\epsilon} > 1+\epsilon.$$

With this, we can check now the same conditions for s_1. If $2s_1 \leq N$, we can apply again this argument of bootstrapping to obtain

$$u \in W^{2,s_2}(\Omega)\ , \quad s_2 = \frac{p_2}{(2^*-2)}\ , \quad p_2 = \frac{Ns_1}{N-2s_1}$$

and we have

$$\frac{s_2}{s_1} = \frac{p_1}{2^*} = \frac{Ns_1(N-2s)}{Ns(N-2s_1)} > (1+\epsilon)\frac{N-2s}{N-2s_1} > 1+\epsilon.$$

Within a finite number of times, we shall obtain $u_k \in W^{2,s'}(\Omega)$, $2s' > N$.

Applying one more time the Sobolev-Morrey Embeddings, we will finally have $u_k \in C^{0,\beta}(\overline{\Omega})$ for $\beta \in (0,1)$. We can equate β and α and this regularity will be then shared by $h(x) = H(u(.),\phi_k(.))$ and again by the Theory of Regularity, we obtain $u_k \in C^2(\overline{\Omega})$.

With that, $f_k(u_k)$ is continuous up to the closure $\overline{\Omega}$ and thus ϕ_k will also belong to $C^2(\overline{\Omega})$ due to the second auxiliary equation. This means that the pair (u_k,ϕ_k) is actually a strong solution to the problem (P_k). Additionally, the Sobolev-Morrey Embeddings will also give us the relation between the two norms

$$\|u\|_{C^{0,\lambda}(\overline{\Omega})} \leq C\,\|u\|_{W^{2,s'}(\overline{\Omega})}\,.$$

We must, however, pay close attention to the fact that this does not give us a uniform limitation on the $C^{0,\lambda}(\overline{\Omega})$ norm of the sequence u_k, since the function h, and therefore the constant limiting $\|u_k\|_{W^{2,s'}(\overline{\Omega})}$, depend on the value of k.

Moreover, the regularity of the sequence (u_k,ϕ_k) is not the final goal of this section. As we have already mentioned, we need a uniform lower bound for u_k to assert that its limit u_λ will be strictly positive in the entire domain and for that, we utilize Lemma 3.1.1. It is easy to see that each u_k will be a supersolution of the problem

$$\begin{cases} -\Delta u = \frac{\lambda}{(u+\frac{1}{k})^\gamma}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.12}$$

since

$$-\Delta u_k = \phi_k(u_k)^{2^*-2} + \frac{\lambda}{(u_k+\frac{1}{k})^\gamma} \geq \frac{\lambda}{(u_k+\frac{1}{k})^\gamma}$$

and since we can easily prove, using the same inequality and the Maximum Principles, that $u_k > 0$ for every $k \in \mathbb{N}$. For a subsolution, we can use the eigenfunction φ_1 of the first eigenvalue of the Laplacian operator $-\Delta$. We know it to be smooth, strictly positive, and bounded in $\overline{\Omega}$ (the unfamiliar reader may look well spread references such as [3] to convince themselves). Furthermore, for every $\delta > 0$, $w = \delta\varphi_1$ is easily seen to be another solution to the eigenvalue problem with λ_1 and it is such that $|w|_\infty = \delta|\varphi_1|_\infty$. Thus, taking δ satisfying

$$\delta\,|\varphi_1|_\infty\,(\delta|\varphi_1|_\infty + 1)^\gamma \leq \frac{\lambda}{\lambda_1},$$

we shall obtain

$$-\Delta w = \lambda_1 w \leq \frac{\lambda}{(\delta|\varphi_1|_\infty + \frac{1}{k})^\gamma} \leq \frac{\lambda}{(w+\frac{1}{k})^\gamma},$$

meaning w is the strictly positive subsolution we were looking for.

Applying Lemma 3.1.1 with $g(s) = \frac{\lambda}{(s+\frac{1}{k})^\gamma}$, $v_1 = w$ and $v_2 = u_k$, we shall have

$$u_k(x) \geq \delta\varphi_1(x) > 0 \quad , \; x \in \Omega \; , \; k \in \mathbb{N}.$$

Supposing that there exist a pointwise limit $u_\lambda \in H_0^1(\Omega)$ to the sequence (u_k) as $k \to \infty$, we then conclude that $u_\lambda \geq \delta\varphi_1 > 0$ a.e. in Ω. This concludes the assertion that $(u_\lambda, \phi_\lambda)$ is a pair of solutions to problem (P_2).

3.2 PROOF OF THEOREM 3.0.1

In this section we finally prove our main result. As mentioned in the last section, we shall need to first prove the existence of solution for a sequence of auxiliary problems, defined, for each $k \in \mathbb{N}$, to be

$$\begin{cases} -\Delta u - \phi u^{2^*-2} = \frac{\lambda}{(u+\frac{1}{k})^\gamma}, & x \in \Omega, \\ -\Delta\phi = f_k(u), & x \in \Omega, \\ u > 0, & x \in \partial\Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases} \qquad (P_k)$$

f_k being the auxiliary functions presented in Section 3.1.

To carry out the process of finding the solution to (P_2), we must first look for a solution to each equation (P_k). For that, we follow a analogous path as the one in the preceding chapter. Let $\beta = \{e_1, e_2, \ldots, e_n, \ldots\}$ be now a orthonormal basis of $H_0^1(\Omega)$ and we define again the subspaces $V_m = [e_1, e_2, \ldots, e_m]$ of $H_0^1(\Omega)$ as being generated by the first m vectors of β, for each $m \in \mathbb{N}$, and equipped with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}}$, the same as the one for the whole space. Again, we can construct an isomorphism from

V_m to $\mathbb{R}^m$ and, therefore, $V_m \times V_m$ will be isomorphic[2] to $\mathbb{R}^{2m}$. That allows us to define the function $\Phi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^{2m}$ whose coordinate functions are

$$\Phi(\zeta,\xi) = (F_1(\zeta,\xi), ..., F_m(\zeta,\xi), G_1(\zeta,\xi), ..., G_m(\zeta,\xi)),$$

$$F_j(\zeta,\xi) = \int_\Omega \nabla u \nabla e_j \, dx - \int_\Omega \phi_+ (u_+)^{2^*-2} e_j \, dx - \lambda \int_\Omega \frac{e_j}{(u_+ + \frac{1}{k})^\gamma} \, dx,$$

$$G_j(\zeta,\xi) = \int_\Omega \nabla \phi \nabla e_j \, dx - \int_\Omega f_k(u_+) e_j \, dx, \quad j = 1, 2, \dots, m,$$

where $u = \sum_{i=1}^m \zeta_i e_i$ and $\phi = \sum_{i=1}^m \xi_i e_i$ are, respectively, the functions in V_m related to the elements $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_m)$ and $\xi = (\xi_1, \xi_2, \dots, \xi_m)$ in $\mathbb{R}^m$ through the isomorphism mentioned above.

Once more, we shall apply Lemma 2.0.1 to obtain weak solutions to each (P_k), so the next step is to show that Φ satisfies its conditions. The continuity of Φ is quite straightforward, so what remains to be done is to prove the following proposition.

Proposition 3.2.1. *The function $\Phi : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ defined above is continuous.*

Proof. This demonstration goes along a similar path from Proposition 2.1.1, from which we can even readily state the continuity of each component function G_j.[3] Now, for F_j, we need only prove that, if a sequence of pairs $((u_n, \phi_n))_{n\in\mathbb{N}} \subset V_m \times V_m$ converges in the $H_0^1(\Omega) \times H_0^1(\Omega)$ norm to (u, ϕ), then

$$\int_\Omega \phi_{n+} (u_{n+})^{2^*-2} e_j \, dx \longrightarrow \int_\Omega \phi_+ (u_+)^{2^*-2} e_j \, dx \tag{3.13}$$

and

$$\int_\Omega \frac{e_j}{(u_{n+} + \frac{1}{k})^\gamma} \, dx \longrightarrow \int_\Omega \frac{e_j}{(u_+ + \frac{1}{k})^\gamma} \, dx. \tag{3.14}$$

For that, let us see that

$$\left|\phi_{n+} (u_{n+})^{2^*-2} e_j\right| \le \frac{2^*-1}{2^*} \left|\phi_{n+}\right|^{\frac{2^*}{2^*-1}} \left|u_{n+}\right|^{2^*\left(\frac{2^*-2}{2^*-1}\right)} + \frac{1}{2^*} \left|e_j\right|^{2^*}$$

and since $2^* - 1$ and $\frac{2^*-1}{2^*-2}$ are conjugate exponents in the Holder sense,

$$\left|\phi_{n+} (u_{n+})^{2^*-2} e_j\right| \le \frac{1}{2^*} \left|\phi_{n+}\right|^{2^*} + \frac{2^*-2}{2^*} \left|u_{n+}\right|^{2^*} + \frac{1}{2^*} \left|e_j\right|^{2^*}.$$

Now, by the continuous embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, the uniform boundness of $\|u_n\|$ and $\|\phi_n\|$ implies the same uniform boundness for $|u_n|_{2^*}$ and $|\phi_n|_{2^*}$, that is, the uniform boundness in $L^1(\Omega)$ of $|u_n|^{2^*}$ and $|\phi_n|^{2^*}$.

---

2 We can consider here the "euclidean" norm in $V_m \times V_m$, where $\|(u,v)\| = \sqrt{\|u\|^2 + \|v\|^2}$, or any equivalent norm in this space. We shall stick to this one, for convenience.

3 Notice that the functions f_k in Proposition 2.1.1 is completely different from the ones we consider now. Nevertheless, the important factor in our proofs are the estimates over each function, which in this case is even stronger than in the last chapter, being bounded in $L^\infty(\Omega)$.

Meanwhile,

$$\left| \frac{e_j}{(u_{n+} + \frac{1}{k})^\gamma} \right| \leq k^\gamma \, |e_j| \, ,$$

meaning both sequences are bounded by functions in $L^1(\Omega)$. Using the DCT, we obtain (3.13) and (3.14). □

Proposition 3.2.2. *There exists a real number $R > 0$ such that, for $\|(\zeta, \xi)\| = R$, we have $\langle \Phi(\zeta, \xi), (\zeta, \xi) \rangle \geq 0$.*

Proof. By definition, we have

$$\begin{aligned}
\langle \Phi(\zeta,\xi), (\zeta,\xi) \rangle &= \sum_{i=1}^m F_i(\zeta,\xi)\zeta_i + \sum_{i=1}^m G_i(\zeta,\xi)\xi_i \\
&= \sum_{i=1}^m \left(\int_\Omega \nabla u \nabla e_i \, dx - \int_\Omega \phi_+ (u_+)^{2^*-2} e_i \, dx - \lambda \int_\Omega \frac{e_i}{(u_+ + \frac{1}{k})^\gamma} \, dx \right) \zeta_i \\
&\quad + \sum_{i=1}^m \left(\int_\Omega \nabla \phi \nabla e_i \, dx - \int_\Omega f_k(u_+) e_i \, dx \right) \xi_i \\
&= \int_\Omega \nabla u \nabla \left(\sum_{i=1}^m e_i \zeta_i \right) \, dx - \int_\Omega \phi_+ (u_+)^{2^*-2} \left(\sum_{i=1}^m e_i \zeta_i \right) \, dx \\
&\quad -\lambda \int_\Omega \frac{(\sum_{i=1}^m e_i \zeta_i)}{(u_+ + \frac{1}{k})^\gamma} \, dx + \int_\Omega \nabla \phi \nabla \left(\sum_{i=1}^m e_i \xi_i \right) \, dx \\
&\quad - \int_\Omega f_k(u_+) \left(\sum_{i=1}^m e_i \zeta_i \right) \, dx \\
&= \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \phi_+ (u_+)^{2^*-1} \, dx - \lambda \int_\Omega \frac{u}{(u_+ + \frac{1}{k})^\gamma} \, dx \\
&\quad + \int_\Omega |\nabla \phi|^2 \, dx - \int_\Omega f_k(u_+) \phi \, dx.
\end{aligned}$$

Thus, we are left with

$$\begin{aligned}
\langle \Phi(\zeta,\xi), (\zeta,\xi) \rangle = \|u\|^2 + \|\phi\|^2 - \int_\Omega \phi_+ (u_+)^{2^*-1} \, dx \\
- \lambda \int_\Omega \frac{u}{(u_+ + \frac{1}{k})^\gamma} \, dx - \int_\Omega f_k(u_+) \phi \, dx.
\end{aligned} \tag{3.15}$$

Using the fact that f_k satisfies the conditions from Lemma 3.1.3 and defining

$$\Omega_k^+ := \{ x \in \Omega \, ; \, |u_+(x)| \geq 1/k \}, \tag{3.16}$$

$$\Omega_k^- := \{ x \in \Omega \, ; \, |u_+(x)| < 1/k \}, \tag{3.17}$$

we have (noting that $f_k(s)$ is positive when $s > 0$)

$$\begin{aligned}\int_\Omega f_k(u_+)\phi\, dx \;&\le\; \int_{\Omega_k^+} L_1\,(u_+)^{2^*-1}\phi\, dx + \int_{\Omega_k^-} L_2\,(u_+)^2\phi\, dx\\ &\le \int_\Omega L_1\,(u_+)^{2^*-1}\phi\, dx + \frac{1}{k^2}\left(\int_{\Omega_k^-} L_2^2\, dx\right)^{1/2} |\phi|_2 \\ &\le \int_\Omega L_1\,(u_+)^{2^*-1}\phi\, dx + \frac{L_2'}{k^2}\left\|\phi\right\|,\end{aligned} \tag{3.18}$$

from which we conclude that

$$\begin{aligned}\int_\Omega \phi_+(u_+)^{2^*-1}\, dx + \int_\Omega \phi f_k(u_+)\, dx \le \int_\Omega \phi_+\left((u_+)^{2^*-1} + L_1(u_+)^{2^*-1}\right)\, dx\\ + L_2'\frac{1}{k^2}\left\|\phi\right\|\\ \le 2L_1'\int_\Omega \phi_+(u_+)^{2^*-1}\, dx + \frac{L_2'}{k^2}\left\|\phi\right\|,\end{aligned} \tag{3.19}$$

where we named $L_1' = \max\{L_1, 1\}$. Now, we will use the Young Inequality, namely

$$ab \le \frac{1}{p}a^p + \frac{1}{p'}b^{p'},$$

where p and p' are conjugated Holder indices, to obtain an estimate of the form

$$\int_\Omega \phi_+(u_+)^{2^*-1}\, dx + \int_\Omega \phi f_k(u_+)\, dx \le D_1\left\|\phi\right\|^{2^*} + D_2\left\|u\right\|^{2^*} + \frac{L_2}{k^2}\left\|\phi\right\|. \tag{3.20}$$

For that, we choose $p = \frac{2^*}{2^*-1}$ so that $p' = 2^*$. With this, we have

$$|\phi||u|^{2^*-1} \le \frac{1}{p'}|\phi|^{p'} + \frac{1}{p}|u|^{(2^*-1)p} = \frac{1}{2^*}|\phi|^{2^*} + \frac{2^*-1}{2^*}|u|^{2^*},$$

so that by the Sobolev Embedding Theorems and inequality (3.19), we obtain estimate (3.20) as intended. Thus, the scalar product satisfies

$$\langle\Phi(\zeta,\xi),(\zeta,\xi)\rangle \ge \left\|u\right\|^2 + \left\|\phi\right\|^2 - D_2\left\|u\right\|^{2^*} - D_1\left\|\phi\right\|^{2^*} - \frac{L_2}{k^2}\left\|\phi\right\| - \lambda C_1(1+\left\|u\right\|),$$

which we can rewrite as

$$\langle\Phi(\zeta,\xi),(\zeta,\xi)\rangle \ge ||(\zeta,\xi)||^2 - D_2\|\zeta\|^{2^*} - D_1\|\xi\|^{2^*} - \frac{L_2}{k^2}\|\xi\| - \lambda C_1(1+\|(\zeta,\xi)\|),$$

where we used

$$||\zeta|| \le \sqrt{||\zeta||^2 + ||\xi||^2} = \|(\zeta,\xi)\|.$$

Thus, choosing R such that

$$R^2 - (D_1+D_2)R^{2^*} > 0,$$

which is equivalent to

$$R^{2^*-2} < \frac{1}{D_1 + D_2},$$

we can take

$$\lambda < \Lambda = \frac{R^2 - (D_1 + D_2)R^{2^*-2}}{C_1(1+t_0)} \quad \text{and} \quad k^2 > \frac{L_2 R}{R^2 - (D_1 + D_2)R^{2^*-2} - \lambda C_1(1+t_0)}$$

and, for all (ζ, ξ) such that $\|(\zeta, \xi)\| = R$, we have

$$\langle \Phi(\zeta, \xi), (\zeta, \xi) \rangle > 0.$$

□

We have then proved the necessary conditions for us to use Lemma 2.0.1, which gives us a pair of sequences of functions $(u_m, \phi_m)_{m \in \mathbb{N}}$, both composed by elements of each V_m, satisfying

$$\int_\Omega \nabla u_m \nabla e_j \, dx - \int_\Omega \phi_{m+} (u_{m+})^{2^*-2} e_j \, dx - \lambda \int_\Omega \frac{e_j}{(u_{m+} + \frac{1}{k})^\gamma} \, dx = 0,$$

$$\int_\Omega \nabla \phi_m \nabla e_j \, dx - \int_\Omega f_k(u_{m+}) e_j \, dx = 0,$$

for $j = 1, 2, ..., m$. Because we are dealing with basis elements, we can expand this to the whole space V_m, so that

$$\int_\Omega \nabla u_m \nabla \omega \, dx - \int_\Omega \phi_{m+} (u_{m+})^{2^*-2} \omega \, dx - \lambda \int_\Omega \frac{\omega}{(u_{m+} + \frac{1}{k})^\gamma} \, dx = 0, \tag{3.21}$$

$$\int_\Omega \nabla \phi_m \nabla \omega \, dx - \int_\Omega f(u_{m+}) \omega \, dx = 0, \quad \omega \in V_m. \tag{3.22}$$

It is important to notice that both sequences satisfy $\|u_m\|, \|\phi_m\| \leq R$ and that this limiting constant does not depend on the index m. We have obtained then a pair of sequences with its terms limited, on $H_0^1(\Omega)$, by a common constant. By known results, namely the Sobolev Embedding Theorems, we can extract a pair of subsequences, which we still denote by $(u_m), (\phi_m)$, and a pair of functions $u, \phi \in H_0^1(\Omega)$ such that

$$\begin{aligned} u_m \rightharpoonup u \text{ in } H_0^1(\Omega) \quad &\text{and} \quad u_m \to u \text{ in } L^s(\Omega), \\ \phi_m \rightharpoonup \phi \text{ in } H_0^1(\Omega) \quad &\text{and} \quad \phi_m \to \phi \text{ in } L^s(\Omega), \quad s \in [1, 2^*). \end{aligned} \tag{3.23}$$

Thus, letting $m \to \infty$ in equations (3.21) and (3.22) and keeping ω in a particular fixed V_l space, we have

$$\int_\Omega \phi_{m+} (u_{m+})^{2^*-2} \omega \, dx \longrightarrow \int_\Omega \phi_+ (u_+)^{2^*-2} \omega \, dx,$$

$$\int_\Omega \frac{\omega}{(u_{m+} + \frac{1}{k})^\gamma} \, dx \longrightarrow \int_\Omega \frac{\omega}{(u_+ + \frac{1}{k})^\gamma} \, dx,$$

$$\int_\Omega f_k(u_{m+})\omega\ dx \longrightarrow \int_\Omega f_k(u_+)\omega\ dx,$$

where the last one can be readily verified by the strong regularity and boundness of each f_k. Using these convergences, we can rewrite

$$\int_\Omega \nabla u \nabla \omega\ dx - \int_\Omega \phi_+(u_+)^{2^*-2}\omega\ dx - \lambda \int_\Omega \frac{\omega}{(u_+ + \frac{1}{k})^\gamma}\ dx = 0,$$

$$\int_\Omega \nabla \phi \nabla \omega\ dx - \int_\Omega f_k(u_+)\omega\ dx = 0, \quad \omega \in V_l.$$

Since here $l \in \mathbb{N}$ is arbitrary, we can pass the limit $l \to \infty$ and achieve

$$\int_\Omega \nabla u \nabla \omega\ dx - \int_\Omega \phi_+(u_+)^{2^*-2}\omega\ dx - \lambda \int_\Omega \frac{\omega}{(u_+ + \frac{1}{k})^\gamma}\ dx = 0, \tag{3.24}$$

$$\int_\Omega \nabla \phi \nabla \omega\ dx - \int_\Omega f_k(u_+)\omega\ dx = 0, \quad \omega \in H_0^1(\Omega). \tag{3.25}$$

Here we used the fact that the V_l spaces are dense in $H_0^1(\Omega)$, which permits us to approximate any test function by elements of V_l.

At last, we can show that $u \geq 0$ for every $x \in \Omega$. This is evident, since taking $\omega = u_- = \max\{-u, 0\}$ in (3.24) leads to

$$-\|u_-\|^2 = \int_\Omega \nabla u \nabla (u_-)\ dx = \int_\Omega \phi_+(u_+)^{2^*-2}(u_-)\ dx + \lambda \int_\Omega \frac{u_-}{(u_+ + \frac{1}{k})^\gamma}\ dx\ \geq\ 0,$$

showing that $u = u_+$. Furthermore, the same argument for (3.25), together with the fact that f satisfies condition (3.1), shows that

$$-\|\phi_-\|^2 = \int_\Omega f_k(u)\phi_-\ dx \geq 0,$$

which implies that $\phi = \phi_+$ and the functions u, ϕ will then be a pair of weak solutions to (P_k). We will from now on denote them by u_k and ϕ_k, to reassure their dependence on the parameter k in the auxiliary system.

Now, the final step to prove Theorem 3.0.1 is to argue that the sequences u_k, ϕ_k tend to functions which satisfy the conditions of weak solutions to problem (P_2). For that, let us notice that, because of the weak convergences in the space $H_0^1(\Omega)$, we have

$$\|u_k\|\ \leq\ \liminf_{m\to\infty} \|u_m\|\ \leq\ R \tag{3.26}$$

and the same applies to each function ϕ_k. Again, the limiting constant does not depend on the index k of the functions of the sequence. That means we are left with new bounded sequences in $H_0^1(\Omega)$ and once more we can affirm that there exists functions $u_\lambda, \phi_\lambda \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\begin{aligned} u_k \rightharpoonup u_\lambda \text{ in } H_0^1(\Omega) \qquad &\text{and} \qquad u_k \to u_\lambda \text{ in } L^s(\Omega), \\ \phi_k \rightharpoonup \phi_\lambda \text{ in } H_0^1(\Omega) \qquad &\text{and} \qquad \phi_k \to \phi_\lambda \text{ in } L^s(\Omega), \quad s \in [1, 2^*). \end{aligned} \tag{3.27}$$

We want to show that, letting $k \to \infty$, we can assert the following convergences

$$\int_\Omega \phi_k (u_k)^{2^*-2} \omega \, dx \longrightarrow \int_\Omega \phi_\lambda (u_\lambda)^{2^*-2} \omega \, dx, \tag{3.28}$$

$$\int_\Omega f_k(u_k) \omega \, dx \longrightarrow \int_\Omega f(u_\lambda) \omega \, dx. \tag{3.29}$$

For that, let us show first that $\phi_k(u_k)^{2^*-2}$ and $f_k(u_k)$ are both bounded sequences in $L^p(\Omega)$, with $p = \frac{2^*}{2^*-1}$. For the latter, it is an easy task because, by condition (3.1),

$$\int_\Omega |f_k(u_k)|^p \ dx \le \int_\Omega |u_k|^{(2^*-1)p} \ dx = |u_k|^{2^*}_{L^{2^*}(\Omega)} \le CR^{2^*}.$$

For (3.28) we only need to make use one more time of the Young Inequality. Now, we can choose $q = 2^* - 1$, which gives $q' = \frac{2^*-1}{2^*-2}$ and $pq'(2^* - 2) = 2^*$. Therefore,

$$\begin{aligned} \int_\Omega \left|\phi_k u_k^{2^*-2}\right|^p \ dx &\le \frac{1}{2^*-1} \int_\Omega |\phi_k|^{2^*} \ dx + \frac{2^*-2}{2^*-1} \int_\Omega |u_k|^{2^*} \ dx \\ &\le \frac{1}{2^*-1} \left\|\phi_k\right\|^{2^*} + \frac{2^*-2}{2^*-1} \left\|u_k\right\|^{2^*}. \end{aligned} \tag{3.30}$$

Now, by the $L^p(\Omega)$ convergence in (3.27), we have $\phi_k(u_k)^{2^*-2} \to \phi_\lambda u_\lambda^{2^*-2}$ a.e. in Ω. In the same manner, using that f is a continuous function, we have $f(u_k) \to f(u)$ a.e. in Ω. By the limitations of $\left|\phi_k u_k^{2^*-2}\right|_p$ and $|f(u_j)|_p$, using Theorem 3.1.1, we obtain $\phi_k(u_k)^{2^*-2} \rightharpoonup \phi_\lambda u_\lambda^{2^*-2}$ and $f(u_k) \rightharpoonup f(u_\lambda)$ in $L^p(\Omega)$.

On the other hand, being p and 2^* conjugate indices in the Holder sense, meaning that $2^* + p = 2^* p$, the integral $\int_\Omega v\omega \ dx$ is finite for $v \in L^p(\Omega)$, since

$$\left|\int_\Omega vw \ dx\right| = |v \ w|_1 \le |v|_p \, |w|_{2^*} < +\infty,$$

where we used the Holder inequality. Therefore, we can define the functional $J(v) = \int_\Omega vw \ dx$ for every function $v \in H_0^1(\Omega)$. The weak convergences we have just obtained imply then (3.28) and (3.29).

As for the limit of the sequence accompanying λ, we have the following: By the developments of Section 3.1, each u_k will be limited from below by $\delta\varphi_1$ and, by the Hardy-Sobolev Inequality (see Appendix), we have $\frac{\omega}{(\varphi_1)^\gamma} \in L^1(\Omega)$, which permits us to use the DCT to conclude that

$$\int_\Omega \frac{\omega}{(u_k + \frac{1}{k})^\gamma} \ dx \longrightarrow \int_\Omega \frac{\omega}{u_\lambda^\gamma} \ dx,$$

since the convergence a.e. of the sequence inside the integral is straightforward. From this, we finally obtain

$$\int_\Omega \nabla u_\lambda \nabla \omega \ dx - \int_\Omega \phi_\lambda (u_\lambda)^{2^*-2} \omega \ dx - \lambda \int_\Omega \frac{\omega}{u_\lambda^\gamma} \ dx = 0,$$

$$\int_{\Omega} \nabla \phi_\lambda \nabla \omega \, dx - \int_{\Omega} f(u_\lambda)\omega \, dx = 0, \quad \omega \in H_0^1(\Omega).$$

The pair $(u_\lambda, \phi_\lambda)$ then satisfies the equations necessary for being a pair of weak solutions of the main problem (P_2). Since the fact that $u_\lambda > 0$ was already proven in Section 3.1, we conclude the assertion that $(u_\lambda, \phi_\lambda)$ is a pair of solutions to problem (P_2) and Theorem 3.0.1 is proven.

4 GENERALIZED SCHRODINGER-POISSON SYSTEM WITH THE N-LAPLACIAN OPERATOR AND CRITICAL EXPONENTIAL GROWTH

Finally, we treat in this chapter our last proposed problem, namely

$$\begin{cases} -\Delta_N u - \phi \frac{f(u)}{u} = \frac{\lambda}{u^\gamma} & \text{in} \quad \Omega, \\ -\Delta_N \phi = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{P_3}$$

We remind the reader that we are imposing over the function f the following exponential critical growth

$$0 \leq f(s)s \leq L|s|^{r+1} \exp\{\alpha s^{\frac{N}{N-1}}\}, \qquad L, \alpha > 0, \quad r > N-1, \tag{4.1}$$

inspired by the Trudinger-Moser inequality (1.10), and that the results we have proven are the following

Theorem 4.0.1. *If $f : [0, \infty) \longrightarrow \mathbb{R}$ is a continuous function satisfying the growth condition (4.1). Then there exists $\Lambda > 0$ such that, for every $0 < \lambda < \Lambda$, problem (P_3) has a solution pair $u_\lambda, \phi_\lambda \in W_0^{1,N}(\Omega)$.*

Meanwhile, treating the alternative problem, where f is of exponential form,

$$\begin{cases} -\Delta_N u + \phi u^{r-1} \exp\{\alpha u^{N'}\} = \frac{\lambda}{u^\gamma} & \text{in} \quad \Omega, \\ -\Delta_N \phi = u^r \exp\{\alpha u^{N'}\} & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P_4}$$

we have

Theorem 4.0.2. *Let us suppose $\alpha > 0$ arbitrary and r such that*

$$(\gamma + rN' - 1)\left(\frac{1-\gamma}{N-1}\right) > 1.$$

Then there exists $\Lambda > 0$ such that, for every $0 < \lambda < \Lambda$, problem (P_4) has a solution pair $u_\lambda, \phi_\lambda \in C_0^1(\overline{\Omega})$. If $\alpha = 0$, the problem (P_4) has a unique positive solution for every $0 < \lambda < \Lambda$.

4.1 PRELIMINARY RESULTS

We have seen in both preceding chapters how important the Fundamental Lemma is to the application of the Galerkin Method. It has, however, an important restriction which we shall need to remove here. Indeed, we see that in the statement of Lemma 2.0.1, the sphere where we find that $\langle h(\alpha), \alpha\rangle \geq 0$ must be generated by the norm arising from this inner product, that is, the euclidean norm in $\mathbb{R}^N$. Results analogous to Propositions 2.1.1 and 3.2.2, concerning Problem (P_3), nonetheless, will enforce us to use a different sphere, formed by an alternative norm in the same space.

This improvement was achieved in [39]. In the following hypothesis, $|\cdot|_e = \sqrt{\langle\cdot,\cdot\rangle}$ is the Euclidean norm on $\mathbb{R}^N$ and $|\cdot|_d$ a general norm.

Lemma 4.1.1. *Let $h : (\mathbb{R}^N, |\cdot|_d) \longrightarrow (\mathbb{R}^N, |\cdot|_d)$ be a continuous function such that $\langle h(\alpha), \alpha\rangle \geq 0$ for every $\alpha \in \mathbb{R}^N$ with $|\alpha|_d = R$, for some $R > 0$. Then there exists an element $z \in \overline{B^d_R(0)} = \{x \in \mathbb{R}^N \ ; \ |x|_d \leq R\}$ such that $h(z) = 0$.*

Proof. Firstly, we know there must exist a constant $c > 0$ such that

$$|x|_d \leq c\,|x|_e\,, \text{ for all } x \in \mathbb{R}^N. \tag{4.2}$$

Now, let us suppose, by contradiction, that $F(x) \neq 0$, for all $x \in \overline{B^d_R(0)}$. We define $f : (\mathbb{R}^N, |\cdot|_d) \longrightarrow (\mathbb{R}^N, |\cdot|_d)$ by

$$f(x) = -\frac{R}{|h(x)|_d} h(x),$$

which, in particular, maps continuously $B^d_R(0)$ into itself. By Brouwer's fixed point theorem, Theorem 2.0.4, there must exist a $z \in \overline{B^d_R(0)}$ such that $f(z) = z$, that is, $|x|_d = R$.

Thus, by hypothesis and using (4.2),

$$0 < R^2 \leq c\,\langle z, z\rangle = c\,\langle f(z), z\rangle = -c\,\frac{R}{|h(x)|_d}\langle h(z), z\rangle \leq 0.$$

This is a contradiction, which concludes our proof. $\square$

It is worthwhile to reassure here the importance of this result. In the methods seen here and throughout several other papers, we are often dealing with Banach spaces, such as $W^{1,N}_0(\Omega)$, where the lack of orthogonality might bring up several problems. The freedom of choice we have with the norm in Lemma 4.1.1 is an efficient way for avoiding such problems.

Following this first result, let us define a property satisfied by the N-Laplacian operator which will be important later.

Definition 4.1.1. If X is a reflexive Banach space and $V\colon X \to X^*$, we say that V is of type (S_+) if, for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ satisfying $x_n \rightharpoonup x$ and[1]

$$\limsup_{n\to+\infty} \langle V(x_n), x_n - x\rangle \leq 0,$$

we obtain $x_n \to x$ in X.

[1] We use here the notation that $\langle w, u\rangle$ symbolizes the action of a dual vector $w \in X^*$ over some element $u \in X$. It should be clear the moments we use this notation, that should not be confused with the exact notion of the inner product over a vector space.

Now, we can take the map $V\colon W_0^{1,N}(\Omega) \to (W_0^{1,N}(\Omega))^*$ given by

$$\langle V(u), v\rangle = \int_\Omega |\nabla u|^{N-2}\,\nabla u \nabla v\, dx \quad \forall\, u, v \in W_0^{1,N}(\Omega).$$

It can be shown that V defined in this way is in fact of type (S_+). In that way, we say that the $-\Delta_N$ operator has the (S_+)-property.

We now dedicate the rest of this section of preliminary results to stating some regularity theorems, which will be quite important later in this work. First, we cite a famous result by Ladyzhenskaya and Ural'tseva, present in their great and accomplished work [40]. Their version contemplates general operators other than the N-Laplacian, but we shall write here only the particular case for simplification.

Theorem 4.1.1. *Let $u \in W^{1,m}(\Omega) \cap L^q(\Omega)$, $m \le N$ and $q \ge \frac{Nm}{N-m}$, be a weak solution of the problem*

$$\begin{cases} -\Delta_N u + a(x, u, \nabla u) = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.3}$$

with $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfying

$$(sign\ u)a(x, u, p) \le (1 + |u|^{\alpha_1})\phi_2(x) + (1 + |u|^{\alpha_2})\phi_2(x)|p|^{m-\epsilon} \tag{4.4}$$

for $\epsilon, \alpha_i, \phi_i$ such that

1. $\frac{N}{N+q} \le \epsilon \le m$;
2. $\phi_i \in L^{r_i}(\Omega), \quad i = 1, 2,$
$$r_1 > \frac{N}{m}\ ; \quad r_2 > \frac{N}{\epsilon}\ ;$$
3. $0 \le \alpha_1 < m\frac{N+q}{N} - 1 - \frac{q}{r_1}$,
$0 \le \alpha_2 < \epsilon\frac{N+q}{N} - 1 - \frac{q}{r_2}$.

Suppose further that $\sup_{x\in\partial\Omega} |u(x)| = M_0 < +\infty$. *Then,* $\max_\Omega |u|$ *is bounded by an expression in terms of* $|u|_{L^q(\Omega)}, M_0, \epsilon, \alpha_i, |\phi|_{L^{r_i}(\Omega)}$.

The next result is due to [41] and gives a strong regularity for a bounded solution to elliptic problems involving the N-Laplacian.

Theorem 4.1.2. *Let α, Λ, M_0 be positive constants with $\alpha \le 1$, Φ be a nonnegative constant and $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,\alpha}$ boundary. Consider the problem*

$$\begin{cases} -\Delta_N u + B(x, u, \nabla u) = 0, & x \in \Omega \\ u = \phi, & x \in \partial\Omega, \end{cases} \tag{4.5}$$

with B satisfying

$$|B(x, z, p)| \le \Lambda(1 + |p|)^{m+2},$$

for all $(x, z, p) \in \partial\Omega \times [-M_0, M_0] \times \mathbb{R}^n$*. If* $\phi \in C^{1,\alpha}(\Omega)$ *is such that* $|\phi|_{1+\alpha} \leq \Phi$ *and if* u *is a bounded weak solution of the Dirichlet problem* (4.5)*, with* $|u| \leq M_0$ *in* Ω*, then there is a positive constant* $\beta = \beta(\alpha, \Lambda, m, n)$ *such that* $u \in C^{1,\beta}(\Omega)$*. Moreover,*

$$|u|_{1+\beta} \leq C(\alpha, \Lambda, m, n, M_0, \Phi, \Omega).$$

The last regularity result appears in [42] and will be important to demonstrate the positiveness of our auxiliary solutions.

Theorem 4.1.3. *Consider the differential inequality*

$$-\Delta_N u + B(x, u, \nabla u) \leq 0 \tag{4.6}$$

in a domain $\Omega \subset \mathbb{R}^n$*. If it holds that*

(I_1) $\quad B(x, z, p) \geq -\kappa\Phi(|p|) - f(z),$

(I_2) $\quad f(0) = 0$ *and* f *is non-decreasing on some interval* $(0, \delta)$, $\delta > 0$,

then the Strong Maximum Principle is valid, meaning that for a non-negative classical solution u *of* (4.6)*, if* $u(x) = 0$ *for some* $x \in \Omega$*, then* $u \equiv 0$*.*

Lastly, we present a comparison result for the N-Laplacian operator, which will be crucial for the proof of the positiveness of the main solution. Recall first that by a subsolution of the problem

$$\begin{cases} -\Delta_N v = g(v), & x \in \Omega \\ v = 0, & x \in \partial\Omega, \end{cases} \tag{4.7}$$

we mean $v_1 \in W_0^{1,N}(\Omega)$ such that $v_1 \geq 0$ a.e. on $\partial\Omega$ and

$$\int_\Omega |\nabla v_1|^{p-1} \nabla v_1 \nabla \omega \, dx \leq \int_\Omega g(v_1)\omega \, dx \;, \quad \forall\, \omega \in W_0^{1,N}(\Omega) \text{ with } \omega \geq 0 \text{ a.e. in } \Omega.$$

Similarly, v_2 is a supersolution of (4.7) if $v_2 \geq 0$ a.e. on $\partial\Omega$ and the reverse inequality above is satisfied, again for $\omega \geq 0$ a.e. in Ω. With this, we state the following lemma, which is again a particular case of a more general result (see [43], where it is considered a problem with the (p,q)-Laplacian operator).

Theorem 4.1.4. *Consider* $g\colon \mathbb{R} \to \mathbb{R}$ *a continuous function satisfying* $t^{1-N}g(t)$ *decreasing for* $t > 0$*. If* u_1, u_2 *are positive sub and supersolution, respectively, of* (4.7)*,* $u_i \in L^\infty(\Omega) \cap C^{1,\alpha}(\Omega)$ *for some* $\alpha \in (0, 1)$*,* $\Delta_N u_i \in L^\infty(\Omega)$ *and* $u_i/u_j \in L^\infty(\Omega)$ *for* $i, j = 1, 2$*, then* $u_2 \geq u_1$ *in* Ω*.*

4.2 AUXILIARY PROBLEMS AND REGULARITY OF THE WEAK SOLUTIONS

In the same way as Chapter 3, Problem (P_3) possesses a singularity in its first equation and, additionally, we now ask that f grows at most exponentially. Like before, these aspects will force us to solve, first, a sequence of auxiliary equations in which we substitute f by more regular (and more importantly, bounded) functions. More specifically, for each $k \in \mathbb{N}$, we shall consider the following system

$$\begin{cases} -\Delta_N u - \phi \frac{f_k(u)}{(u+\frac{1}{k})} = \frac{\lambda}{(u+\frac{1}{k})^\gamma}, & x \in \Omega, \\ -\Delta_N \phi = f_k(u), & x \in \Omega, \\ u > 0, & x \in \partial\Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases} \tag{P_k}$$

where f_k is the same Strauss sequence encountered in (3.5). They are given - we recall - by

$$f_k(s) = \begin{cases} -k[G(-k-\frac{1}{k}) - G(-k)], & if \quad s \leq -k \\ -k[G(s-\frac{1}{k}) - G(s)], & if \quad -k \leq s \leq -\frac{1}{k} \\ k^2 s[G(-\frac{2}{k}) - G(-\frac{1}{k})], & if \quad -\frac{1}{k} \leq s \leq 0 \\ k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})], & if \quad 0 \leq s \leq \frac{1}{k} \\ k[G(s+\frac{1}{k}) - G(s)], & if \quad \frac{1}{k} \leq s \leq k \\ k[G(k+\frac{1}{k}) - G(k)], & if \quad s \geq k, \end{cases} \tag{4.8}$$

with $G(s) = \int_0^s f(\xi) d\xi$. Another fact we can recall from Chapter 3 is Lemma 3.1.2, which states conclusions of regularity and convergence for f_k. Even further, we can adapt our proof of Lemma 3.1.3, found in the same chapter, now that f satisfies (4.1). For an outline of the proof of this adapted result, see [39].

Lemma 4.2.1. *The sequence of auxiliary functions f_k defined above satisfies*

1. $\forall\, k \in \mathbb{N},\ 0 \leq s\ f_k(s) \leq C_1 |s|^{r+1}\ exp\{2^{N'} \alpha s^{N'}\}, \quad |s| \geq \frac{1}{k},$

2. $\forall\, k \in \mathbb{N},\ 0 \leq s\ f_k(s) \leq C_2 |s|^2\ exp\{2^{N'} \alpha s^{N'}\}, \quad |s| \leq \frac{1}{k},$

with C_1, C_2 being two positive constants independent of the parameter k.

What we intend to do, eventually, is to prove the existence of the sequence (u_k, ϕ_k), solutions to each (P_k), and subsequently show that we can obtain a pair $(u_\lambda, \phi_\lambda)$, the limit of a subsequence of (u_k, ϕ_k), which satisfies the condition for being weak solutions of the main problem (P_3). We remind that this last fact is characterized by the equalities

$$\int_\Omega |\nabla^{N-2} u| \nabla u \nabla \omega\, dx - \int_\Omega \phi \frac{f(u)}{u} \omega\, dx - \lambda \int_\Omega \frac{\omega}{u^\gamma}\, dx = 0,$$

$$\int_\Omega |\nabla^{N-2} \phi| \nabla \phi \nabla \omega\, dx - \int_\Omega f(u) \omega\, dx = 0, \quad \omega \in W_0^{1,N}(\Omega).$$

The regularity of each pair (u_k, ϕ_k) is then a crucial factor to the final result, as is their sign in Ω. This is also the reason we must consider first the auxiliary functions f_k. Their regularity implies quite directly the strong regularity of each solution ϕ_k, which in turn does the same for u_k, as we shall see ahead. Furthermore, it is known that even though each u_k is strictly positive, the same might not be said for its limit u. We will be able, nonetheless, to obtain the existence of a uniform lower bound for u_k, which easily translates to a lower bound of its limit.

Therefore, we shall present in this section the regularity of each pair (u_k, ϕ_k) of weak solutions to the auxiliary problem (P_k). The existence of the limits $(u_\lambda, \phi_\lambda)$, here only required to be a.e. limits, will be already assumed and we will prove it in further sections.

Proposition 4.2.1. *If (u_k, ϕ_k) is a pair of non-negative weak solutions of the auxiliary problem (P_k), then it is a pair of classical solutions. Furthermore, there exists a strictly positive lower bound $w \in L^\infty(\Omega)$ for the sequence u_k, i. e., w is such that*

$$u_k \geq w > 0\ , \quad \forall\, k \in \mathbb{N}.$$

Proof. Firstly, we must refer to Theorem 4.1.1 to show that $\phi_k \in L^\infty(\Omega)$. We notice the importance, here, of using the auxiliary functions f_k instead of f, which does not satisfy the proper conditions for the application of Theorem 4.1.1. Next, using this, we can also induce the following estimate for the nonlinearity of the first equation

$$\left|\phi_k \frac{f_k(u_k)}{(u_k+\frac{1}{k})} + \frac{\lambda}{(u_k+\frac{1}{k})^\gamma}\right| \leq |\phi_k|_\infty \left|\frac{f_k(u_k)}{u_k}\right| + \lambda k^\gamma \leq c_k|\phi_k|_\infty + \lambda k^\gamma.$$

Then, the same theorem is again applicable and gives us $u_k \in L^\infty(\Omega)$. For both of this functions, we can now apply Theorem 4.1.2 to ensure that $\phi_k, u_k \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ (the indexes may be different at first, but we remember we can always consider the smaller of the two). At last, Theorem 4.1.3 states that we can apply the strong maximum principles for both equations and thus, together with the results $\phi_k, u_k \geq 0$ and $\phi_k, u_k \neq 0$, which we have already verified, gives us $\phi_k, u_k > 0$ in Ω. We have, finally, obtained that u_k, ϕ_k are classical solutions to (P_k).

Moreover, the regularity of the sequence (u_k, ϕ_k) is not the final goal of this section. As we have already mentioned, we need a uniform lower bound for u_k to assert that its limit u_λ will be strictly positive in the entire domain and for that, we utilizes Theorem 4.1.4. It is easy to notice that each u_k will be a supersolution of the problem (4.7) considering $g(s) = \frac{\lambda}{(s+\frac{1}{k})^\gamma}$, since

$$-\Delta_N u_k = \phi_k (u_k)^{2^*-2} + \frac{\lambda}{(u_k+\frac{1}{k})^\gamma} \geq \frac{\lambda}{(u_k+\frac{1}{k})^\gamma}$$

and we have just proven that $u_k > 0$ in Ω. For a subsolution, we use the eigenfunction of the first eigenvalue of the Laplacian operator $-\Delta$. We know it to be smooth, strictly positive, and bounded in $\overline{\Omega}$. Not only that, but we can also obtain a constant $B > 0$ such that $|\Delta_N \varphi_1| \leq B$ in Ω. Furthermore, for every $\delta > 0$, we have

$$-\Delta_N(\delta\varphi_1) = -\delta^{N-1}\Delta_N\varphi_1 \leq \delta^{N-1}B$$

and if we denote by $w = \delta\varphi_1$, then w will be such that $|w|_\infty = \delta|\varphi_1|_\infty$. Thus, taking δ satisfying

$$\delta^{N-1}B(\delta|\varphi_1|_\infty + 1)^\gamma \leq \lambda,$$

we shall obtain

$$-\Delta_N w = \delta^{N-1}B \leq \frac{\lambda}{(\delta|\varphi_1|_\infty + \frac{1}{k})^\gamma} \leq \frac{\lambda}{(w + \frac{1}{k})^\gamma},$$

meaning w is the strictly positive subsolution we were looking for. We only need now to prove that w and u_k satisfy the conditions necessary to apply Theorem 4.1.4. We have already proved them to be in $L^\infty(\Omega) \cap C^{1,\beta}(\Omega)$ and this implies quite directly in $\Delta_N w, \Delta_N u_k \in L^\infty(\Omega)$.

What is left then for us to verify is that $u_k/w, w/u_k \in L^\infty(\Omega)$. For any compact contained in Ω, this fact is evident since both are positive continuous functions. By that, we need to show now that, when $x \to \partial\Omega$ (assuming, of course, $x \in \Omega$), we have

$$\max\left\{\limsup_{x\to\partial\Omega}\frac{u_k}{w}, \limsup_{x\to\partial\Omega}\frac{w}{u_k}\right\} < +\infty \tag{4.9}$$

and for that, we apply a boundary point lemma. This result can be seen in [42] and assumes the same conditions as Theorem 4.1.3, so that we have no problems to apply the result. It then states that using the Hopf boundary point lemma, we obtain

$$\frac{\partial u_k}{\partial\nu}(x_0) < 0\ , \quad \frac{\partial w}{\partial\nu}(x_0) < 0\ , \quad x_0 \in \partial\Omega, \tag{4.10}$$

where ν is the exterior normal unit vector to $\partial\Omega$. Thus, both expressions in (4.10), together with l'Hôpital's theorem, imply condition (4.9). Finally, we have achieved all of Theorem 4.1.4 conditions, and using it we obtain that $u_k \geq w > 0$ for every $k \in \mathbb{N}$. $\square$

From this, if $u_k \to u_\lambda$ a.e. in Ω, then we can conclude that $u_\lambda > 0$ in Ω.

4.3 PROOF OF THEOREM 4.0.1

In this section, we finally prove our main result. As mentioned in the last section, we shall need to first prove the existence of solution for a sequence of auxiliary problems, defined, for each $k \in \mathbb{N}$, to be

$$\begin{cases} -\Delta_N u - \phi\frac{f_k(u)}{(u+\frac{1}{k})} = \frac{\lambda}{(u+\frac{1}{k})^\gamma}, & x \in \Omega, \\ -\Delta_N\phi = f_k(u), & x \in \Omega, \\ u > 0, & x \in \partial\Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases} \tag{P_k}$$

f_k being the auxiliary functions presented in Section 4.2.

Only after obtaining proper solutions to (P_k) will we be able to find a solution pair to (P_3). For that, we shall take the limit of these auxiliary solutions and prove both the convergence and the affirmation that their limit satisfies (P_3). As is the case for the application of the Galerkin method, we start by taking a Schauder basis $\mathcal{B} = \{e_1, e_2, \ldots, e_n, \ldots\}$ of $W_0^{1,N}(\Omega)$ and with it we define the subspaces $V_m = [e_1, e_2, \ldots, e_m]$ of $W_0^{1,N}(\Omega)$ spanned by the first m vectors of $\mathcal{B}$. Let us cite more explicitly the isomorphism between $\mathbb{R}^m$ and V_m. Again we shall work with the Cartesian space $V_m \times V_m$. For some $(\xi, \zeta) = (\xi_1, \cdots, \xi_m, \zeta_1, \cdots, \zeta_m) \in \mathbb{R}^{2m}$, the quantity

$$|(\xi, \zeta)|_m = \left(\left| \sum_{j=1}^{m} \xi_j e_j \right|^N_{W_0^{1,N}(\Omega)} + \left| \sum_{j=1}^{m} \zeta_j e_j \right|^N_{W_0^{1,N}(\Omega)} \right)^{1/N}$$

is a norm in $\mathbb{R}^{2m}$, which can be directly sen from the properties of the norm $|\cdot|_{W_0^{1,N}(\Omega)}$, and thus equivalent to the euclidean norm in the same space. In this manner, we identify the spaces $V_m \times V_m$ and $\mathbb{R}^{2m}$, using the equivalence

$$(\xi, \zeta) = (\xi_1, \cdots, \xi_m, \zeta_1, \cdots, \zeta_m) \in \mathbb{R}^{2m} \longleftrightarrow (u, \phi) = \left(\sum_{j=1}^{m} \xi_j e_j, \sum_{j=1}^{m} \zeta_j e_j \right) \in V_m.$$

We are thus in position to define the function $\Phi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^{2m}$ whose coordinate functions are

$$\Phi(\zeta, \xi) = (F_1(\zeta, \xi), ..., F_m(\zeta, \xi), G_1(\zeta, \xi), ..., G_m(\zeta, \xi)),$$

$$F_j(\zeta, \xi) = \int_\Omega |\nabla u|^{N-2} \nabla u \nabla e_j \, dx - \int_\Omega \phi_+ \frac{f_k(u_+)}{(u + \frac{1}{k})} \, dx - \lambda \int_\Omega \frac{e_j}{(u_+ + \frac{1}{k})^\gamma} \, dx,$$

$$G_j(\zeta, \xi) = \int_\Omega |\nabla \phi|^{N-2} \nabla \phi \nabla e_j \, dx - \int_\Omega f(u_+) e_j \, dx,$$

where $j = 1, 2, \ldots, m$, $u = \sum_{i=1}^m \zeta_i e_i$ and $\phi = \sum_{i=1}^m \xi_i e_i$ are the functions in V_m related to the elements $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m)$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ in $\mathbb{R}^m$ through the isomorphism mentioned above.

To get the desired results, we shall also use Lemma 4.1.1, so our next step is to show that Φ satisfies its conditions. The continuity of Φ is quite straightforward, meaning we only need to prove the following proposition.

Proposition 4.3.1. *There exists a real number $\rho > 0$ and a norm $|\cdot|_d$ such that, for $|(\zeta, \xi)|_d = \rho$, we have $\langle \Phi(\zeta, \xi), (\zeta, \xi) \rangle \geq 0$.*

Proof. By definition, we have

$$\begin{aligned}
\langle \Phi(\zeta,\xi),(\zeta,\xi)\rangle &= \sum_{i=1}^{m} F_i(\zeta,\xi)\zeta_i + \sum_{i=1}^{m} G_i(\zeta,\xi)\xi_i \\
&= \sum_{i=1}^{m}\left(\int_\Omega |\nabla u|^{N-2}\nabla u \nabla e_i \, dx \right.\\
&\quad \left. - \int_\Omega \phi_+ \frac{f_k(u_+)}{(u+\frac{1}{k})} e_j \, dx - \lambda \int_\Omega \frac{e_j}{(u_+ + \frac{1}{k})^\gamma}\, dx\right)\zeta_i \\
&\quad + \sum_{i=1}^{m}\left(\int_\Omega |\nabla \phi|^{N-2}\nabla \phi \nabla e_i \, dx - \int_\Omega f_k(u_+) e_i \, dx\right)\xi_i \\
&= \int_\Omega |\nabla u|^{N-2}\nabla u \nabla\left(\sum_{i=1}^{m} e_i\zeta_i\right) dx - \int_\Omega \phi_+ \frac{f_k(u_+)}{(u+\frac{1}{k})}\left(\sum_{i=1}^{m} e_i\zeta_i\right) dx \\
&\quad - \lambda \int_\Omega \frac{(\sum_{i=1}^{m} e_i\zeta_i)}{(u_+ + \frac{1}{k})^\gamma}\, dx + \int_\Omega |\nabla \phi|^{N-2}\nabla\phi\nabla\left(\sum_{i=1}^{m} e_i\xi_i\right) dx \\
&\quad - \int_\Omega f_k(u_+)\left(\sum_{i=1}^{m} e_i\zeta_i\right) dx \\
&= \int_\Omega |\nabla u|^N \, dx - \int_\Omega \phi_+ \frac{f_k(u_+)u_+}{(u+\frac{1}{k})}\, dx - \lambda \int_\Omega \frac{u}{(u_+ + \frac{1}{k})^\gamma}\, dx \\
&\quad + \int_\Omega |\nabla\phi|^N \, dx - \int_\Omega f_k(u_+)\phi \, dx.
\end{aligned}$$

Therefore, we are left with

$$\begin{aligned}
\langle \Phi(\zeta,\xi),(\zeta,\xi)\rangle = \|u\|^N + \|\phi\|^N - \int_\Omega \phi_+ \frac{f_k(u_+)u_+}{(u+\frac{1}{k})}\, dx \\
- \lambda \int_\Omega \frac{u}{(u_+ + \frac{1}{k})^\gamma}\, dx - \int_\Omega f_k(u_+)\phi \, dx.
\end{aligned} \tag{4.11}$$

Using the fact that $f(s)$ is positive when $s > 0$, we have

$$\int_\Omega \phi_+ \frac{f_k(u_+)u_+}{(u+\frac{1}{k})}\, dx + \int_\Omega \phi f(u_+)\, dx \le 2\int_\Omega \phi_+ f(u_+)\, dx. \tag{4.12}$$

To obtain an estimate of this term, we first recall, from (3.16) and (3.17) of Chapter 3, the sets Ω_k^+ and Ω_k^-. With this, we can separate the integral from (4.12) and we shall have

$$\int_\Omega f_k(u_+)\phi \, dx = \int_{\Omega_k^+} f_k(u_+)\phi \, dx + \int_{\Omega_k^-} f_k(u_+)\phi \, dx.$$

By Lemma 4.2.1, we have the following

$$\begin{aligned}
\left|\int_{\Omega_k^+} f_k(u_+)\phi \, dx\right| &\le \int_{\Omega_k^+} |u_+|^r|\phi| \, exp\{2^{N'}\alpha u^{N'}\}\, dx \\
&\le \left[\int_{\Omega_k^+} |u_+|^{N'r}|\phi|^{N'}\, dx\right]^{1/N'} \left[\int_{\Omega_k^+} exp\{N2^{N'}\alpha|u_+|^{N'}\}\, dx\right]^{1/N},
\end{aligned} \tag{4.13}$$

where we have used Holder Inequality with the exponents N and $N' = \frac{N}{N-1}$. Using the same result for the first term, we obtain

$$\int_\Omega |u_+|^{N'r}|\phi|^{N'}\ dx \leq \left(\int_\Omega |u|^{N'(r+1)}\ dx\right)^{\frac{r}{r+1}} \left(\int_\Omega |\phi|^{N'(r+1)}\ dx\right)^{\frac{1}{r+1}},$$

since

$$\frac{r}{r+1} + \frac{1}{r+1} = 1.$$

Besides that, if $\rho \geq \|u\|_{W_0^{1,N}(\Omega)}$, then

$$\int_\Omega exp\{N2^{N'}\alpha|u|^{N'}\}\ dx \leq \int_\Omega exp\left\{N2^{N'}\alpha\rho^{N'}\left(\frac{|u|}{\|u\|_{W_0^{1,N}(\Omega)}}\right)^{N'}\right\}\ dx$$

and, for $\rho \leq \frac{1}{2}\left(\frac{\alpha_N}{N\alpha}\right)^{N'}$, we have, by the Trudinger-Moser Inequality,

$$\int_\Omega exp\{N2^{N'}\alpha|u|^{N'}\}\ dx \leq L^{\frac{1}{N}}(N)|\Omega|^{\frac{1}{N}}. \tag{4.14}$$

Therefore, we have just obtained the estimate

$$\left|\int_{\Omega_k^+} f_k(u_+)\ dx\phi\right| \leq \|u\|^r\,\|\phi\|\,(L(N)|\Omega|)^{\frac{1}{N}}. \tag{4.15}$$

This is just half of the solution. Now, we need to estimate the integral of $\phi f_k(u_+)$ on Ω_k^- and again by using the result of Lemma 4.2.1, we have

$$\begin{aligned}\left|\int_{\Omega_k^-} f_k(u_+)\phi\ dx\right| &\leq C_2\int_{\Omega_k^-}|u_+|^2 exp\{2^{N'}\alpha u_+^{N'}\}|\phi|\ dx \\ &\leq \frac{1}{k^2}\left[\int_{\Omega_k^-}|\phi|^{N'}\ dx\right]^{1/N'}\left[\int_{\Omega_k^-} exp\{N2^{N'}\alpha|u_+|^{N'}\}\ dx\right]^{1/N}.\end{aligned} \tag{4.16}$$

Using (4.14) one more time, we have

$$\left|\int_{\Omega_k^-} f_k(u_+)\phi\ dx\right| \leq \frac{1}{k^2}\,\|\phi\|\,(L(N)|\Omega|)^{\frac{1}{N}}. \tag{4.17}$$

Combining (4.15) and (4.17), we finally obtain

$$\left|\int_\Omega f_k(u_+)\phi\ dx\right| \leq \left(\|u\|^r\,\|\phi\| + \frac{1}{k^2}\,\|\phi\|\right)(L(N)|\Omega|)^{\frac{1}{N}},$$

which, together with (4.11), implies that

$$\langle\Phi(\zeta,\xi),(\zeta,\xi)\rangle \geq \|u\|^N + \|\phi\|^N - \left(\|u\|^r\,\|\phi\| + \frac{1}{k^2}\,\|\phi\|\right)(L(N)|\Omega|)^{\frac{1}{N}} - \lambda\int_\Omega \frac{u}{(u_+ + \frac{1}{k})^\gamma}\ dx.$$

The last term can also be estimated by the norm $\|u\|$, now by the application of the Sobolev Embedding Theorems,

$$\left| \int_\Omega \frac{u}{(u_+ + \frac{1}{k})^\gamma}\, dx \right| \leq \int_\Omega u^{1-\gamma}\, dx \leq \int_\Omega (u+1)\, dx \leq C\,\|u\| + |\Omega|,$$

where we have used $1-\gamma \in (0,1)$.

Thus, we obtain

$$\langle \Phi(\zeta,\xi), (\zeta,\xi)\rangle \geq \|u\|^N + \|\phi\|^N - \left(\|u\|^r \|\phi\| + \frac{1}{k^2}\|\phi\|\right)(L(N)|\Omega|)^{\frac{1}{N}} - \lambda(C\,\|u\| + |\Omega|).$$

To complete our proof, we need to find $\rho > 0$ such that $\langle \Phi(\zeta,\xi), (\zeta,\xi)\rangle \geq 0$ for $|(\zeta,\xi)|_d = \rho$, $|\cdot|_d$ being a norm on $\mathbb{R}^{2N}$. We shall choose here

$$|(\zeta,\xi)|_d^N = |\zeta|_e^N + |\xi|_e^N,$$

where $|\zeta|_e$ is the Euclidean norm on $\mathbb{R}^m$, which is equal to the norm $\|u\|$ of $u \in V_m$ image of $\zeta \in \mathbb{R}^N$ by the isomorphism between the two spaces (and the same reasoning for $|\xi|_e$). Notice that with this definition, we have

$$\|u\| \leq |(\zeta,\xi)|_d\,, \qquad \|\phi\| \leq |(\zeta,\xi)|_d$$

and with this, we can rewrite

$$\begin{aligned} \langle \Phi(\zeta,\xi), (\zeta,\xi)\rangle \geq |(\zeta,\xi)|_d^N &- (L(N)|\Omega|)^{\frac{1}{N}}\, |(\zeta,\xi)|_d^{r+1} \\ &- \frac{1}{k^2}\|\phi\|\,(L(N)|\Omega|)^{\frac{1}{N}} - \lambda(C\,\|u\| + |\Omega|). \end{aligned} \tag{4.18}$$

Remembering we still need ρ be small enough so that (4.14) is satisfied, we take

$$\rho < \min\left\{ \frac{1}{2}\left(\frac{\alpha_N}{\alpha N}\right)^{N'}, (L(N)|\Omega|)^{\frac{N-(r+1)}{N}} \right\},$$

which implies

$$|(\zeta,\xi)|_d^N - (L(N)|\Omega|)^{\frac{1}{N}}\,|(\zeta,\xi)|_d^{r+1} > 0.$$

Taking $k \in \mathbb{N}$ and $\lambda > 0$ such that

$$\rho^N - (L(N)|\Omega|)^{\frac{1}{N}}\rho^{r+1} > \frac{(L(N)|\Omega|)^{\frac{1}{N}}\rho}{k^2}$$

and

$$\rho^N - (L(N)|\Omega|)^{\frac{1}{N}}\rho^{r+1} - \frac{(L(N)|\Omega|)^{\frac{1}{N}}\rho}{k^2} > \lambda(C\rho + |\Omega|),$$

we obtain, for all (ζ,ξ) such that $|(\zeta,\xi)|_d = \rho$,

$$\langle \Phi(\zeta,\xi), (\zeta,\xi)\rangle > 0,$$

as we wanted.

$\square$

We then proved the necessity for us to use Lemma 4.1.1, which gives us a pair of sequences of functions u_m, ϕ_m, both composed by elements of each V_m, satisfying

$$\int_\Omega |\nabla u_m|^{N-2}\nabla u_m \nabla e_j \, dx - \int_\Omega \phi_{m+}\frac{f_k(u_{m+})}{u_{m+}+\frac{1}{k}} e_j \, dx - \lambda \int_\Omega \frac{e_j}{(u_{m+}+\frac{1}{k})^\gamma} \, dx = 0,$$

$$\int_\Omega |\nabla \phi_m|^{N-2}\nabla \phi_m \nabla e_j \, dx - \int_\Omega f_k(u_{m+}) e_j \, dx = 0$$

for $j = 1, 2, ..., m$. Because we are dealing with basis elements, we can expand this to the whole space V_m, so that

$$\int_\Omega |\nabla u_m|^{N-2}\nabla u_m \nabla \omega \, dx - \int_\Omega \phi_{m+}\frac{f_k(u_{m+})}{u_{m+}+\frac{1}{k}} \omega \, dx - \lambda \int_\Omega \frac{\omega}{(u_{m+}+\frac{1}{k})^\gamma} \, dx = 0, \tag{4.19}$$

$$\int_\Omega |\nabla \phi_m|^{N-2}\nabla \phi_m \nabla \omega \, dx - \int_\Omega f_k(u_{m+}) \omega \, dx = 0, \quad \omega \in V_m. \tag{4.20}$$

It is important to notice that both sequences satisfy $\|u_m\|, \|\phi_m\| \leq \rho$ and that this limiting constant does not depend on the index m. We have obtained then a pair of sequences with its terms limited, on $W_0^{1,N}(\Omega)$, by a common constant. By known results, namely the Sobolev Embedding Theorems, we can extract a pair of subsequences, which we still denote by $(u_m), (\phi_m)$, and a pair of functions $u, \phi \in W_0^{1,N}(\Omega)$ such that

$$\begin{aligned} u_m \rightharpoonup u \text{ in } W_0^{1,N}(\Omega) \qquad &\text{and} \qquad u_m \to u \text{ in } L^s(\Omega), \\ \phi_m \rightharpoonup \phi \text{ in } W_0^{1,N}(\Omega) \qquad &\text{and} \qquad \phi_m \to \phi \text{ in } L^s(\Omega), \quad s \in [N, \infty). \end{aligned} \tag{4.21}$$

Notice that this should also imply convergence in $L^s(\Omega)$, with $s \in [1, N)$, since Ω is of finite measure.

What we show now is that we can actually assert the strong convergences in $W_0^{1,N}(\Omega)$,

$$u_m \to u \text{ in } W_0^{1,N}(\Omega) \qquad \text{and} \qquad \phi_m \to \phi \text{ in } W_0^{1,N}(\Omega). \tag{4.22}$$

For that, let us first consider two sequences, provided to us by the fact that $\mathcal{B} = \{e_1, e_2, \cdots, e_n, \cdots\}$ is a Schauder basis, $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $(\beta_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$u = \sum_{i=1}^{\infty} \alpha_i e_i \qquad \text{and} \qquad \phi = \sum_{i=1}^{\infty} \alpha_i e_i \tag{4.23}$$

and thus

$$\psi_n = \sum_{i=1}^{n} \alpha_i e_i \to u \qquad \text{and} \qquad \theta_n = \sum_{i=1}^{n} \beta_i e_i \to \phi \text{ in } W_0^{1,N}(\Omega).$$

If we use $(u_m - \psi_m)$ as a test function in (4.19), we shall have

$$\int_\Omega |\nabla u_m|^{N-2}\nabla u_m \nabla(u_m - \psi_m)\, dx - \int_\Omega \phi_{m+}\frac{f_k(u_{m+})}{u_{m+}+\frac{1}{k}}(u_m - \psi_m)\, dx \\ - \lambda \int_\Omega \frac{(u_m-\psi_m)}{(u_{m+}+\frac{1}{k})^\gamma}\, dx = 0. \tag{4.24}$$

Now, since we are dealing with Lipschitz continuous functions, we have, applying Lemma 4.2.1,

$$\begin{aligned}\left|\int_\Omega \phi_{m+}\frac{f_k(u_{m+})}{u_{m+}+\frac{1}{k}}(u_m-\psi_m)\, dx\right| &\leq \int_\Omega |\phi_{m+}|\frac{|f_k(u_{m+})|}{|u_{m+}|}|u_m-\psi_m|\, dx \\ &\leq c_k \int_\Omega |\phi_{m+}||u_m-\psi_m|\, dx \\ &\leq c_k|\phi_{m+}|_{N'}^{N'}|u_m-\psi_m|_N^N.\end{aligned} \tag{4.25}$$

Besides that, we also have

$$\left|\int_\Omega \frac{(u_m-\psi_m)}{(u_{m+}+\frac{1}{k})^\gamma}\, dx\right| \leq k^\gamma |u_m-\psi_m|_1.$$

Thus, by the characterization in (4.23), which asserts $L^s(\Omega)$ convergence for ψ_m, we obtain

$$\lim_{m\to\infty}\int_\Omega |\nabla u_m|^{N-2}\nabla u_m\nabla(u-\psi_m)\, dx = 0,$$

which in turn gives us, together with (4.24) and the above estimates,

$$\lim_{m\to\infty}\int_\Omega |\nabla u_m|^{N-2}\nabla u_m\nabla(u_m-u)\, dx = 0.$$

With this, we have shown that we are able to apply the (S_+)-property of the $-\Delta_N$ operator (see the paragraph right after Definition 4.1.1) and, by doing so, we finally obtain the first claim in (4.22). We shall not repeat our arguments here to not prolong too much our work, but it is evident that the same development can be applied to show the strong convergence of ϕ_m in $W_0^{1,N}(\Omega)$. The only difference is that for that we would use the following estimate

$$\left|\int_\Omega f_k(u_{m+})(\phi_m-\theta_m)\, dx\right| \leq C_k\|u_{m+}\|\|\phi_m-\theta_m\|.$$

Now, going back to equations (4.19) and (4.20), we note that we can take $\omega \in V_l$ for $l \leq n$ and because of that, applying the limit $m \to \infty$ gives us, by (4.22),

$$\int_\Omega |\nabla u|^{N-2}\nabla u\nabla\omega\, dx - \int_\Omega \phi\frac{f_k(u)}{(u+\frac{1}{k})}\omega\, dx - \lambda\int_\Omega \frac{\omega}{(u+\frac{1}{k})^\gamma}\, dx = 0, \tag{4.26}$$

$$\int_\Omega |\nabla\phi|^{N-2}\nabla\phi\nabla\omega\, dx - \int_\Omega f_k(u)\omega\, dx = 0, \quad \omega\in V_l, \tag{4.27}$$

for all $l \in \mathbb{N}$. Since the union of all V_l is dense in $W_0^{1,N}(\Omega)$, we achieve

$$\int_\Omega |\nabla u|^{N-2}\nabla u \nabla \omega \, dx - \int_\Omega \phi \frac{f_k(u)}{(u+\frac{1}{k})}\omega \, dx - \lambda \int_\Omega \frac{\omega}{(u+\frac{1}{k})^\gamma} \, dx = 0, \tag{4.28}$$

$$\int_\Omega |\nabla \phi|^{N-2}\nabla \phi \nabla \omega \, dx - \int_\Omega f_k(u)\omega \, dx = 0, \quad \omega \in W_0^{1,N}(\Omega). \tag{4.29}$$

To prove that these are solutions to (P_k), we must show that $u \geq 0$ for every $x \in \Omega$. For this, we take $\omega = u_- = \max\{-u, 0\}$ in (4.28),

$$\begin{aligned} -\|u_-\|^N &= \int_\Omega |\nabla u|^{N-2}\nabla u \nabla (u_-) \, dx \\ &= \int_\Omega \phi_+ \frac{f_k(u_+)}{u_+ + \frac{1}{k}}(u_-) \, dx + \lambda \int_\Omega \frac{u_-}{(u_+ + \frac{1}{k})^\gamma} \, dx \;\geq\; 0, \end{aligned} \tag{4.30}$$

showing that $u = u_+$. Furthermore, the same argument for (4.29), together with the fact that f satisfies condition (4.1), shows that

$$-\|\phi_-\|^N = \int_\Omega f_k(u_+)\phi_- \, dx \geq 0$$

which implies that $\phi = \phi_+$ and therefore shows that the functions u, ϕ will constitute a pair of weak solutions to (P_k). Also, by the developments of Section 4.2, we know that $u, \phi \in W_0^{1,N}(\Omega) \cap C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. We will from now on denote them by u_k and ϕ_k, to reassure their dependence on the parameter k in the auxiliary system.

Now, the last step to prove Theorem 4.0.1 is to argue that the sequences u_k, ϕ_k tend to functions which satisfy the conditions of weak solutions to Problem (P_3). For that, let us notice that, because of the weak convergences in the space $W_0^{1,N}(\Omega)$, we have

$$\|u_k\| \;\leq\; \liminf_{m\to\infty} \|u_m\| \;\leq\; \rho \tag{4.31}$$

and the same applies to each function ϕ_k. Again, the limiting constant does not depend on the index k of the functions of the sequence. That means we are left with new bounded sequences in $W_0^{1,N}(\Omega)$ and once more we can affirm that there exist functions $u_\lambda, \phi_\lambda \in W_0^{1,N}(\Omega)$ such that, up to a subsequence,

$$\begin{aligned} u_k \rightharpoonup u_\lambda \text{ in } W_0^{1,N}(\Omega) \qquad &\text{and} \qquad u_k \to u_\lambda \text{ in } L^s(\Omega), \\ \phi_k \rightharpoonup \phi_\lambda \text{ in } W_0^{1,N}(\Omega) \qquad &\text{and} \qquad \phi_k \to \phi_\lambda \text{ in } L^s(\Omega), \quad s \in [N, \infty). \end{aligned} \tag{4.32}$$

Once more, we must show, in a manner similar to what we did in Chapter 3 to prove the convergences (3.28) and (3.29), that we have the following, as $k \to +\infty$,

$$\int_\Omega \phi_k \frac{f_k(u_k)}{(u_k + \frac{1}{k})}\omega \, dx \longrightarrow \int_\Omega \phi \frac{f(u_\lambda)}{u_\lambda}\omega \, dx, \tag{4.33}$$

and

$$\int_\Omega f_k(u_k)\omega\, dx \longrightarrow \int_\Omega f(u_\lambda)\omega\, dx. \tag{4.34}$$

Since now the test function ω belongs to $L^N(\Omega)$, we must bound uniformly the norms $\left|\phi_k \frac{f_k(u_k)}{(u_k+\frac{1}{k})}\right|_{N'}$ and $|f_k(u_k)|_{N'}$, where once more $N' = \frac{N}{N-1}$ is the conjugate exponent to N. What we shall do, however, is to bound its norm in $\frac{N}{2}$, so that, given the known estimate (see the Appendix)

$$|h|_{N'} \leq |\Omega|^r\, |h|_{N/2}\ , \quad \text{for } h \in L^{N/2}(\Omega) \text{ and } r = \frac{1}{N'} - \frac{2}{N},$$

we will get the desired result.[2]

Therefore, let us compute

$$\begin{aligned}
\int_\Omega \left|\phi_k \frac{f_k(u_k)}{(u_k+\frac{1}{k})}\right|^{N/2} dx \leq & \left(\int_{\Omega_k^+} |\phi_k|^{N/2}|u_k|^{\frac{N}{2}(r-1)} exp\{2^{N'}(N/2)\alpha|u_k|^{N'}\}\, dx + \right.\\
& \left. + \int_{\Omega_k^-} |\phi_k|^{N/2}\, exp\{2^{N'}(N/2)\alpha|u_k|^{N'}\}\, dx\right)\\
& \leq \left(\int_\Omega |\phi_k|^N |u_k|^{N(r-1)}\, dx\right)^{\frac{1}{2}} \left(\int_\Omega exp\{N2^{N'}\alpha|u|^{N'}\}\, dx\right)^{\frac{1}{2}} + \\
& + \left(\int_\Omega |\phi_k|^N\ dx\right)^{\frac{1}{2}} \left(\int_\Omega exp\{N2^{N'}\alpha|u|^{N'}\}\, dx\right)^{\frac{1}{2}}\\
& \leq (L(N)|\Omega|)^{\frac{1}{2N}} \left[\left(\int_\Omega |\phi_k|^{Nr}\, dx\right)^{\frac{1}{2r}} \left(\int_\Omega |u_k|^{Nr}\, dx\right)^{\frac{r-1}{2r}} + \right.\\
& \left. + \left(\int_\Omega |\phi_k|^N\ dx\right)^{\frac{1}{2}}\right],
\end{aligned} \tag{4.35}$$

where, in the second inequality, we have used (4.14).[3] Thus, by the Sobolev Embeddings, we obtain

$$\int_\Omega \left|\phi_k \frac{f_k(u_k)}{(u_k+\frac{1}{k})}\right|^{N/2} dx \leq \tilde{C}_1 \|\phi_k\|^{N/2} \|u_k\|^{N/2} + \tilde{C}_2 \|\phi_k\|^{N/2} \leq \tilde{C}_1\rho^N + \tilde{C}_2\rho^{N/2}.$$

In a similar way, we can write

$$\begin{aligned}
\int_\Omega |f_k(u_k)|^{\frac{N}{2}}\ dx \leq & \left(\int_{\Omega_k^+} |u_k|^{\frac{N}{2}r} exp\{2^{N'}(N/2)\alpha|u_k|^{N'}\}\, dx + \right.\\
& \left. + \int_{\Omega_k^-} |u_k|^{\frac{N}{2}}\, exp\{2^{N'}(N/2)\alpha|u_k|^{N'}\}\, dx\right)\\
& \leq (L(N)|\Omega|)^{\frac{1}{2N}} \left[\left(\int_\Omega |u_k|^{Nr}\, dx\right)^{\frac{1}{2}} + \left(\int_\Omega |u_k|^N\ dx\right)^{\frac{1}{2}}\right],
\end{aligned} \tag{4.36}$$

[2] Notice that, since $N \geq 3$, we shall have $\frac{N}{2} \geq N'$.

[3] It becomes clear now why we have chosen to use the norm in $L^{\frac{N}{2}}(\Omega)$, rather than in $L^{N'}(\Omega)$. With the latter, it would not be possible to use (4.14), as we have just done.

from which we now obtain

$$\int_\Omega |f_k(u_k)|^{\frac{N}{2}}\ dx \leq \tilde{C}_3 \|u_k\|^{\frac{Nr}{2}} + \tilde{C}_4 \|u_k\|^{\frac{N}{2}} \leq \tilde{C}_3 \rho^{\frac{Nr}{2}} + \tilde{C}_4 \rho^{\frac{N}{2}}.$$

Again by the $L^N(\Omega)$ convergences assured now by (4.32) (and also taking into account the uniform convergence assured by Lemma 3.1.2), we have $\phi_k \frac{f_k(u_k)}{(u_k+\frac{1}{k})} \to \phi_\lambda \frac{f(u_\lambda)}{u_\lambda}$ and $f_k(u_k) \to f(u_\lambda)$ a.e. in Ω. Using Theorem 3.1.1, as we have already done in Chapter 3, we obtain $\phi_k (u_k)^{2^*-2} \rightharpoonup \phi_\lambda u_\lambda^{2^*-2}$ and $f(u_k) \rightharpoonup f(u_\lambda)$ in $L^{N'}(\Omega)$.

We can then define the functional in $L^{N'}(\Omega)$ which relates, to any $v \in L^{N'}(\Omega)$, the number $\int_\Omega v\omega\ dx$, which is well defined, since $\omega \in L^N(\Omega)$. We can, at last, use the weak convergences achieved in the last paragraph and show that , which is equivalent to (4.33) and (4.34).

As for the limit of the sequence accompanying λ, we have the following: By the developments of Section 4.2, each u_k will be limited from below by $\delta\varphi_1$ and, by the Hardy-Sobolev Inequality (see Appendix), we have

$$\frac{\omega}{(\varphi_1)^\gamma} \in L^1(\Omega), \text{ for } \omega \in W_0^{1,N}(\Omega),$$

which permits us to use the DCT to conclude that

$$\int_\Omega \frac{\omega}{(u_k + \frac{1}{k})^\gamma}\ dx \longrightarrow \int_\Omega \frac{\omega}{u_\lambda^\gamma}\ dx,$$

since the convergence a.e. of the sequence inside the integral is straightforward. From this, we finally obtain

$$\int_\Omega |\nabla u_\lambda|^{N-2} \nabla u_\lambda \nabla \omega\ dx - \int_\Omega \phi_\lambda (u_\lambda)^{2^*-2} \omega\ dx - \lambda \int_\Omega \frac{\omega}{u_\lambda^\gamma}\ dx = 0,$$

$$\int_\Omega |\nabla \phi_\lambda|^{N-2} \nabla \phi_\lambda \nabla \omega\ dx - \int_\Omega f(u_\lambda)\omega\ dx = 0, \quad \omega \in H_0^1(\Omega).$$

The pair $(u_\lambda, \phi_\lambda)$ then satisfies the equations necessary for being a pair of weak solutions of the main Problem (P_3). Since the regularity of this pair of functions, as well as the fact that $u_\lambda > 0$, was already proven in Section 4.2, we conclude the assertion that $(u_\lambda, \phi_\lambda)$ is a solution pair to problem (P_3) and Theorem 4.0.1 is proven.

4.4 PROOF OF THEOREM 4.0.2 BY SCHAUDER FIXED POINT THEORY

In this last section, we aim to solve problem (P_4). As we have mentioned, this will be done by way of the Schauder Fixed Point Theorem. For that, let us recall such. For more details and proof, see [44, Corollary 11.2].

Theorem 4.4.1 (Schauder's Fixed Point Theorem). *Let E be a Banach space, and let C be a nonempty closed and convex set in E. Suppose further that $F : C \to C$ is a continuous and compact map, that is, such that $F(C) \subset K$, where $K \subset C$ is a compact subset. Then, F has a fixed point.*

We shall need then to define an operator in the appropriate function space such that a fixed point is the solution to our system. The details will become clearer ahead. First, we introduce some known results from the study of the p-Laplacian and problems concerning such operator.

Let $\phi_0 \in W_0^{1,N}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ be the solution of the torsion problem

$$\begin{cases} -\Delta_N \phi_0 = 1 & \text{in } \Omega \\ \phi_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.37}$$

It follows from [45, 41, 46] that, for $v \in L^\infty(\Omega)$, the equation $-\Delta_p u = v$ in Ω with $u = 0$ on $\partial\Omega$ has a unique weak solution u which belongs to $C^{1,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$ and that the associated solution operator $(-\Delta_p)^{-1} : L^\infty(\Omega) \to C^1(\overline{\Omega})$ is positive, continuous and compact. Moreover, if $v \geq 0$ and $v \not\equiv 0$, then u belongs to the interior of the positive cone in $C^1(\overline{\Omega})$, that is, $u > 0$. Hence $\partial u / \partial \eta < 0$ on $\partial\Omega$ and u is bounded from above and from below by positive multiples of the distance function $\text{dist}(x, \partial\Omega)$. Here η is the unit normal vector to $\partial\Omega$ pointing outwards. Thus $(-\Delta_p)^{-1}$ is a strongly positive operator on $C(\overline{\Omega})$, i.e., $v \in P$ implies $(-\Delta_p)^{-1} v \in int(P)$, where P denotes the cone of positive functions belonging $C(\overline{\Omega})$.

In addition, for the p-Laplacian operator, we can state the following comparison principle, which will be important ahead.

Theorem 4.4.2. *If Ω is a bounded domain in $\mathbb{R}^N$ and if $u, v \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ with $1 < p < \infty$ satisfy, in the weak sense, $-\Delta_p u \leq -\Delta_p v$ on Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

Proof. First of all, we have, by hypothesis, that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \omega \, dx \leq \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \omega \, dx,$$

for every non-negative $\omega \in W_0^{1,p}(\Omega)$. Thus, taking $\omega = \max\{u(x) - v(x), 0\} \in W_0^{1,p}(\Omega)$, the following inequality is satisfied

$$\int_A \left\{ |\nabla u|^{p-2} \nabla u \nabla (u - v) - |\nabla v|^{p-2} \nabla v \nabla (u - v) \right\} \, dx \leq 0, \tag{4.38}$$

where $A = \{x \in \Omega \; ; \; v(x) < u(x)\}$.

On the other hand, we affirm that the following inequality holds for all pair of vectors $a, b \in \mathbb{R}^p$ (if $p > 1$)

$$\left\langle |b|^{p-2} b - |a|^{p-2} a, b - a \right\rangle \geq 0, \tag{4.39}$$

where the equality is satisfied if, and only if, $a = b$.

Indeed, suppose relation (4.39) is not valid. This would be equivalent to saying that

$$|b|^p - |b|^{p-2}\langle b, a\rangle + |a|^p - |a|^{p-2}\langle b, a\rangle < 0,$$

where we have just distributed all the included terms. Therefore,

$$\begin{aligned} |b|^p + |a|^p < (|b|^{p-2} + |a|^{p-2})\langle b, a\rangle &\leq (|b|^{p-2} + |a|^{p-2})\,|b|\,|a| \\ &= (|b|^{p-1}\,|a| + |a|^{p-1}\,|b|), \end{aligned} \tag{4.40}$$

which can also be written as

$$0 < (|b|^{p-1} - |a|^{p-1})(|a| - |b|),$$

which is a contradiction if $p > 1$. For the proof that the equality in (4.39) implies $a = b$, one must do the same reasoning we have just conducted, from which it will be found that $0 \leq (|b|^{p-1} - |a|^{p-1})(|a| - |b|)$, this being only satisfied when $|a| = |b|$. One must then return to (4.39) and concludes the affirmation.

With this, we can return to (4.38), which can only be satisfied if $\nabla u = \nabla v$ a.e. in Ω, since its integrand is always non-negative. This means that $u - v$ must be a constant in A but, since $u = v$ at ∂A and this is a continuous function, we must have A a null measure set, so that $v \geq u$ a.e. in Ω. $\square$

Now, it follows from our considerations in Remark 1.4.2 the existence of a solution $U \in C^1(\overline{\Omega})$ to the problem[4]

$$\begin{cases} -\Delta_N u = \frac{1}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.41}$$

Let us then define

$$U_\infty = \sup_{x \in \Omega} U(x)\ , \qquad \phi_{0,\infty} = \sup_{x \in \Omega} \phi_0(x) \tag{4.42}$$

and consider

$$\begin{aligned} \Lambda = \min \Bigg\{ 1, \left(\frac{1}{U_\infty 2^{\frac{1}{\gamma}}}\right)^{\frac{1}{\frac{1-\gamma}{N-1} + \frac{N-2}{\gamma}}}, \\ \left(\frac{1}{2\phi_{0,\infty} U_\infty^{(rN'+\gamma-1)} \exp\{\alpha U_\infty^{N'} N'\}}\right)^{\frac{1}{(rN'+\gamma-1)\frac{1-\gamma}{N-1} - 1}} \Bigg\}. \end{aligned} \tag{4.43}$$

Furthermore, define at last the operator

$$\begin{aligned} T_\epsilon \colon (C(\overline{\Omega}))^2 &\longrightarrow (C(\overline{\Omega}))^2 \\ (v, \psi) &\longmapsto T_\epsilon(v, \psi) := (u_\epsilon, \phi_\epsilon), \end{aligned} \tag{4.44}$$

---

[4] Indeed, Problem (4.41) is nothing but a decoupled version of System P_3, where we take $f \equiv 0$. Thus, the condition in Remark 1.4.2 is trivially satisfied.

where $(u_\epsilon, \phi_\epsilon)$ is the unique weak solution of

$$\begin{cases} -\Delta_N u = \frac{\lambda}{v^{\gamma+\epsilon}} - \psi v^{r-1}\exp\{\alpha v^{N'}\} & \text{in} \quad \Omega, \\ -\Delta_N \phi = v^r \exp\{\alpha v^{N'}\} & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.45}$$

Given the regularity of both v and ψ, we can affirm, by the comments made above, that T_ϵ is indeed well defined for every $\epsilon > 0$. Now, if we wish to obtain a solution pair $(u, \phi) \in (C(\overline{\Omega}))^2$ such that $u > 0$, we must restrict the domain of T_ϵ to a class of functions with this restriction. For that, we use the solution U to Problem (4.41), together with the function ϕ_0, which will serve as a bound for ϕ. More specifically, we shall consider as the domain of T_ϵ the subset

$$\mathcal{A} = \left\{(v, \psi) \in (C(\overline{\Omega}))^2 : \lambda U \le v \le k_1 \text{ and } 0 \le \psi \le k_2\phi_0\right\} \tag{4.46}$$

of $(C(\overline{\Omega}))^2$, where the positive constants k_1 and k_2 are given in the following

Lemma 4.4.1. *Consider Λ satisfying (4.43), $0 < \lambda < \Lambda$ and $\mathcal{A}$ as defined in (4.46).*

There exists $\epsilon^* > 0$ and $k_1 > 0$ such that, if we suppose that $k_2 > 0$ satisfies

$$k_1^r \exp\{\alpha k_1^{N'}\} \le k_2^{N-1} \tag{4.47}$$

and

$$k_2\phi_{0,\infty}k_1^{\gamma+r-1}\exp\{\alpha k_1^{N'}\} < \frac{\lambda}{2}, \tag{4.48}$$

where we remember r is such that

$$(\gamma + rN' - 1)\left(\frac{1-\gamma}{N-1}\right) > 1, \tag{4.49}$$

then T_ϵ, for $0 < \epsilon < \epsilon^*$, is well defined and T_ϵ maps $\mathcal{A}$ into $\mathcal{A}$.

Proof. First of all, notice that conditions (4.47) and (4.48) are equivalent to

$$k_1^{\frac{r}{N-1}}\exp\{\frac{\alpha}{N-1}k_1^{N'}\} \le k_2 \le \frac{\lambda}{2\phi_{0,\infty}k_1^{\gamma+r-1}\exp\{\alpha k_1^{N'}\}},$$

which is possible if

$$k_1^{(rN'+\gamma-1)}\exp\{\alpha N' k_1^{N'}\} < \frac{\lambda}{2\phi_{0,\infty}}, \tag{4.50}$$

where we have used $\frac{1}{N-1} + 1 = \frac{N}{N-1} = N'$.

Now, by (4.43), we have $\lambda < 1$ and, besides,

$$\lambda U_\infty < \lambda^{\frac{1-\gamma}{N-1}}U_\infty < \left(\frac{1}{2\lambda^{N-2}}\right)^{\frac{1}{\gamma}}.$$

Furthermore, again by (4.43), λ satisfies

$$\lambda^{(rN'+\gamma-1)\frac{1-\gamma}{N-1}} U_\infty^{(rN'+\gamma-1)} \exp\{\alpha U_\infty^{N'} N'\} < \frac{\lambda}{2\phi_{0,\infty}}, \tag{4.51}$$

reminding also that $r > 0$ satisfies (4.49). Let us choose $k_1 > 0$ satisfying

$$\lambda U_\infty < \lambda^{\frac{1-\gamma}{N-1}} U_\infty = k_1 < \left(\frac{1}{2\lambda^{N-2}}\right)^{\frac{1}{\gamma}}. \tag{4.52}$$

With our choice in (4.52) and with condition (4.51), inequality (4.50) will be satisfied and we can thus justify the choices in (4.47) and (4.48). Furthermore, from the first inequality in (4.52), we obtain that $\mathcal{A} \neq \emptyset$.

Moreover, since $\frac{\lambda}{v^\gamma+\epsilon} - \psi v^{r-1}\exp\{\alpha v^{N'}\}, v^r \exp\{\alpha v^{N'}\} \in L^\infty(\Omega)$, system (4.45), as describe before, has an unique solution $(u_\epsilon, \phi_\epsilon)$, showing that the operator T_ϵ is indeed well defined. Let $(v, \psi) \in \mathcal{A}$, then

$$\begin{aligned} -\Delta_N u &= \frac{\lambda}{v^\gamma + \epsilon} - \psi v^{r-1} \exp\{\alpha v^{N'}\} \\ &\geq \frac{\lambda}{k_1^\gamma + \epsilon} - \psi k_1^{r-1} \exp\{\alpha k_1^{N'}\} \\ &\geq \frac{\lambda}{k_1^\gamma + \epsilon} - k_2 \phi_{0,\infty} k_1^{r-1} \exp\{\alpha k_1^{N'}\}. \end{aligned} \tag{4.53}$$

On the other hand, notice that, by (4.48),

$$\lambda - k_2 \phi_{0,\infty} k_1^{\gamma+r-1} \exp\{\alpha k_1^{N'}\} > \frac{\lambda}{2}$$

so that, using (4.52),

$$\frac{\lambda}{k_1^\gamma} - k_2 \phi_{0,\infty} k_1^{r-1} \exp\{\alpha k_1^{N'}\} > \frac{\lambda}{2k_1^\gamma} > \lambda^{N-1}. \tag{4.54}$$

Now, defining the continuous function $G_\lambda : [0, +\infty) \to \mathbb{R}$ by

$$G_\lambda(\epsilon) = \frac{\lambda}{k_1^\gamma + \epsilon} - k_2 \phi_{0,\infty} k_1^{r-1} \exp\{\alpha k_1^{N'}\},$$

we obtain $G_\lambda(0) = \frac{\lambda}{k_1^\gamma} - k_2 \phi_{0,\infty} k_1^{r-1} \exp\{\alpha k_1^{N'}\} > \frac{\lambda}{2k_1^\gamma} > \lambda^{N-1}$ and thus, by continuity, there exists $\epsilon^* = \epsilon^*(\lambda) > 0$ such that,

$$G_\lambda(\epsilon) > \lambda^{N-1} \text{ if } \epsilon \in (0, \epsilon^*).$$

By (4.54) and the definition of $G_\lambda(\epsilon)$, we conclude that

$$-\Delta_N u = G_\lambda(\epsilon) \geq \lambda^{N-1} = -\Delta_N(\lambda \phi_0).$$

By the comparison principle, we obtain

$$\lambda \phi_0 \leq u. \tag{4.55}$$

Now,

$$\begin{aligned} -\Delta_N u &= \frac{\lambda}{v^\gamma + \epsilon} - \psi v^{r-1} \exp\{\alpha v^{N'}\} \\ &\le \frac{\lambda}{v^\gamma} \\ &\le \lambda^{1-\gamma} \frac{1}{U^\gamma} = -\lambda^{1-\gamma} \Delta_N U = -\Delta_N (\lambda^{\frac{1-\gamma}{N-1}} U). \end{aligned} \tag{4.56}$$

By comparison principle and (4.52), we can write

$$u \le \lambda^{\frac{1-\gamma}{N-1}} U \le k_1. \tag{4.57}$$

On the other hand, for ϕ, given as the solution to

$$-\Delta_N \phi = v^r \exp\{\alpha v^{N'}\} \ge 0$$

we obtain, again by the comparison principle, that $\phi \ge 0$.

Furthermore,

$$-\Delta_N \phi = v^r \exp\{\alpha v^{N'}\} \le k_1^r \exp\{\alpha k_1^{N'}\} \le k_2^{N-1} = -\Delta_N (k_2 \phi_0),$$

which implies at last

$$\phi \le k_2 \phi_0. \tag{4.58}$$

Therefore, T maps $\mathcal{A}$ into $\mathcal{A}$, thus completing the proof of Lemma 4.4.1.

□

We are now in the right position to prove, via the Schauder Fixed Point Theorem, our main result, namely Theorem 4.0.2. Let us start with the existence affirmation. We have just seen that Lemma 4.4.1 allows us to define the operator $T_\epsilon : \mathcal{A} \to \mathcal{A}$ given by (4.44) and its continuity can be seen through standard estimates of the regularity theory and the strong notion of convergence we have in $\mathcal{A}$. Notice further that $\mathcal{A}$ is closed and convex. Therefore, remains only to prove that the map T_ϵ is compact. Indeed, considering system (4.45) and defining

$$\Gamma = \begin{pmatrix} \frac{\lambda}{v^\gamma + \epsilon} - \psi v^{r-1} \exp\{\alpha v^{N'}\} \\ v^r \exp\{\alpha v^{N'}\} \end{pmatrix},$$

we have that Γ belongs to $(C(\overline{\Omega}))^2$, which implies that $\Gamma \in (L^p(\Omega))^2$ for any $1 < p < \infty$. By using elliptic estimates [38], we get $T_\epsilon(v, \psi) \in (W^{2,p}(\Omega))^2$, for any $1 < p < \infty$. The Sobolev-Morrey's Embedding Theorem entails $T_\epsilon(v, \psi) \in (C^{1,\rho}(\Omega))^2$, for any $0 < \rho < 1$. Using that $C^{1,\rho}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$, this implies that T_ϵ is compact.

Finally, using Schauder Fixed Point Theorem (Theorem 4.4.1), we get the existence of a fixed point $(u_\epsilon, \phi_\epsilon) \in (C^{1,\rho}(\Omega))^2$ of T_ϵ, that is,

$$\begin{cases} -\Delta_N u_\epsilon = \frac{\lambda}{u_\epsilon^\gamma + \epsilon} - \phi_\epsilon u_\epsilon^{r-1} \exp\{\alpha u_\epsilon^{N'}\} & \text{in} \quad \Omega, \\ -\Delta_N \phi_\epsilon = u_\epsilon^r \exp\{\alpha u_\epsilon^{N'}\} & \text{in} \quad \Omega, \\ u_\epsilon > 0 & \text{in} \quad \Omega, \\ u_\epsilon = \phi_\epsilon = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.59}$$

By compactness results, analogous to what we have done in the previous chapters, we can extract a convergent subsequences in $C^1(\overline{\Omega})$, which we will continue denoting by (u_ϵ) and (ϕ_ϵ), respectively, and $(u, \phi) \in (C^1(\overline{\Omega}))^2$ such that $(u_\epsilon, \phi_\epsilon) \to (u, \phi)$ in the $(C^1(\overline{\Omega}))^2$ topology. Since $(u_\epsilon, \phi_\epsilon) \in \mathcal{A}$, there exist k_1 and k_2, independent of ϵ, such that

$$\lambda U \leq u_\epsilon \leq k_1 \text{ and } 0 \leq \phi_\epsilon \leq k_2 \phi_0. \tag{4.60}$$

By the uniform convergence in weak formulation of (4.59) and (4.60), we get

$$\begin{cases} -\Delta_N u + \phi u^{r-1} \exp\{\alpha u^{N'}\} = \frac{\lambda}{u^\gamma} & \text{in} \quad \Omega, \\ -\Delta_N \phi = u^r \exp\{\alpha u^{N'}\} & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Therefore, according to our construction, we have a weak solution $(u, \phi) \in (C^1(\overline{\Omega}))^2$ and this completes the proof of the existence.

At last, let us prove uniqueness of solution of system (P_4), supposing that $\alpha = 0$. Assume that function pairs (u, ϕ_u) and (v, ϕ_v) are two different positive solutions of system (P_4). Then, using $u - v$ as the test function in the weak formulation of the problem,

$$\int_\Omega |\nabla u|^{N-2}(\nabla u, \nabla(u-v))\, dx + \int_\Omega \phi_u u^{r-1}(u-v)\, dx - \lambda \int_\Omega u^{-\gamma}(u-v)\, dx = 0 \tag{4.61}$$

$$\int_\Omega |\nabla v|^{N-2}(\nabla v, \nabla(u-v))\, dx + \int_\Omega \phi_v v^{r-1}(u-v)\, dx - \lambda \int_\Omega v^{-\gamma}(u-v)\, dx = 0. \tag{4.62}$$

Subtracting (4.62) from (4.61), one obtains

$$\begin{aligned} C\|u-v\|^N &+ \int_\Omega \left[\phi_u u^{r-1} - \phi_v v^{r-1}\right](u-v)\, dx \\ &- \lambda \int_\Omega \left(u^{-\gamma} - v^{-\gamma}\right)(u-v)\, dx \\ &\leq \int_\Omega (|\nabla u|^{N-2}\nabla u - |\nabla v|^{N-2}\nabla v, \nabla(u-v))\, dx \\ &+ \int_\Omega \left[\phi_u u^{r-1} - \phi_v v^{r-1}\right](u-v)\, dx - \lambda \int_\Omega \left(u^{-\gamma} - v^{-\gamma}\right)(u-v)\, dx \\ &= 0. \end{aligned} \tag{4.63}$$

Note that

$$\begin{aligned}
&\int_{\Omega}\left[\phi_u u^{r-1}-\phi_v v^{r-1}\right](u-v)\,dx\\
=&\int_{\Omega}\phi_u u^r\,dx+\int_{\Omega}\phi_v v^r\,dx-\int_{\Omega}\phi_u u^{r-1}v\,dx-\int_{\Omega}\phi_v v^{r-1}u\,dx.
\end{aligned}$$

By the Young inequality, it follows

$$u^{r-1}v\le\frac{r-1}{r}u^r+\frac{1}{r}v^r,\ v^{r-1}u\le\frac{r-1}{r}v^r+\frac{1}{r}u^r.$$

From the above information, there holds

$$\begin{aligned}
&\int_{\Omega}\left[\phi_u u^{r-1}-\phi_v v^{r-1}\right](u-v)\,dx\\
\ge&\frac{1}{r}\left[\int_{\Omega}\left(\phi_u u^r+\phi_v v^r-\phi_u v^r-\phi_v u^r\right)\,dx\right]\\
=&\frac{1}{r}\int_{\Omega}\left(\phi_u-\phi_v\right)\left(u^r-v^r\right)\,dx.
\end{aligned}\tag{4.64}$$

By the definitions of ϕ_u,ϕ_v in (P_4), we have

$$\begin{cases}-\Delta_N\phi_u+\Delta_N\phi_v=u^r-v^r, & \text{in }\Omega\\ \phi_u=\phi_v=0, & \text{on }\partial\Omega.\end{cases}$$

Consequently

$$\begin{aligned}
C\left\|\phi_u-\phi_v\right\|^N\le&\int_{\Omega}(|\nabla\phi_u|^{N-2}\nabla\phi_u-|\nabla\phi_v|^{N-2}\nabla\phi_v,\nabla(\phi_u-\phi_v))\,dx\\
=&\int_{\Omega}\left(\phi_u-\phi_v\right)\left(u^r-v^r\right)\,dx.
\end{aligned}\tag{4.65}$$

Therefore, by (4.64), we deduce that

$$\int_{\Omega}\left[\phi_u u^{r-1}-\phi_v v^{r-1}\right](u-v)\,dx\ge\frac{C}{r}\left\|\phi_u-\phi_v\right\|^N.$$

Since $0<\gamma<1$, we have the following elementary inequality

$$\left(a^{-\gamma}-b^{-\gamma}\right)(a-b)\le 0.$$

Thus, $\int_{\Omega}\left(u^{-\gamma}-v^{-\gamma}\right)(u-v)\,dx\le 0$. Consequently, it follows from (4.62) that

$$C\|u-v\|^N+\frac{C}{r}\left\|\phi_u-\phi_v\right\|^N-\lambda\int_{\Omega}\left(u^{-\gamma}-v^{-\gamma}\right)(u-v)\,dx\le 0.$$

Consequently, $\|u-v\|^N\le 0$ and $\left\|\phi_u-\phi_v\right\|^N\le 0$. This leads to $\|u-v\|^N=0$ and $\left\|\phi_u-\phi_v\right\|^N=0$, which implies that $u(x)=v(x)$ and $\phi_u(x)=\phi_v(x)$ in Ω. So the function pair (u,ϕ_u) is the unique positive solution of system (P_4) when $\alpha=0$. The proof is complete.

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APPENDIX A – SOME IMPORTANT RESULTS

Let us see here some results concerning $L^p(\Omega)$ spaces and Sobolev spaces which are used extensively throughout the present work. We begin with a crucial and well known result for estimating the integral of the product of two functions.

Theorem A.0.1 (Holder Inequality). *Let $p \in (1, +\infty)$ and $p' = \frac{p}{p-1}$ its conjugate exponent, meaning $\frac{1}{p} + \frac{1}{p'} = 1$. Then, if $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, $fg \in L^1(\Omega)$ and*

$$|fg|_1 = \int_\Omega |fg| \ dx \leq \left(\int_\Omega |f| \ dx\right)^{1/p} \left(\int_\Omega |g| \ dx\right)^{1/p'} = |f|_p \, |g|_{p'} \, .$$

Proof. See, for example, [2, Theorem 4.6] □

From Theorem A.0.1, we can also extract an interesting result concerning the relation between different $L^p(\Omega)$ spaces.

Corollary A.0.1.1. *If $p \in [1, +\infty)$, $f \in L^p(\Omega)$ and $r \in [1, p]$, then $f \in L^r(\Omega)$ and*

$$|f|_r \leq |\Omega|^s \, |f|_p \, ,$$

where $s = \frac{1}{r} - \frac{1}{p}$.

Proof. For that, simply use the estimate found in Theorem A.0.1, taking $|f|^r \in L^{\frac{p}{r}}(\Omega)$ and $g \equiv 1 \in L^t(\Omega)$, where t is the conjugate exponent of $\frac{p}{r}$, that is,

$$t = \frac{p/r}{p/r - 1} = \frac{p}{p-r}.$$

Therefore,

$$\begin{aligned} \int_\Omega |f|^r \ dx = ||f|^r \, g|_1 &\leq \left(\int_\Omega |f|^{r\frac{p}{r}} \ dx\right)^{r/p} \left(\int_\Omega |g|^t \ dx\right)^{1/t} \\ &= \left(\int_\Omega |f|^r \ dx\right)^{r/p} \left(\int_\Omega 1 \ dx\right)^{1/t}, \end{aligned} \tag{A.1}$$

which implies

$$|f|_r \leq |\Omega|^{1/(rt)} \, |f|_p \, ,$$

with $\frac{1}{rt} = \frac{1}{r} - \frac{1}{p} = s$. □

Next, we present an estimate concerning functions in the Sobolev Space $W_0^{1,p}(\Omega)$ which helps us deal with singular terms in our problems and prove the convergence of certain integrals. For a example of its importance, see for example [47].

Theorem A.0.2 (Hardy-Sobolev Inequality)**.** *Given $u \in W_0^{1,p}(\Omega)$, $p \in (1, N]$ and $\tau \in [0, 1]$, then $\frac{u}{\varphi^\tau} \in L^r(\Omega)$, where φ_1 is an eigenfunction of $(-\Delta, H_0^1(\Omega))$ associated with the first eigenvalue $\lambda_1 > 0$ and $r > 0$ is such that $\frac{1}{r} = \frac{1}{p} - \frac{1-\tau}{N}$. Moreover, there exists $C > 0$ such that*

$$\left|\frac{u}{\varphi_1^\tau}\right|_r \leq C \left|\nabla u\right|_p .$$

Proof. See [48]. □

Remark A.0.1. Notice that, in Theorem A.0.2, since $r \geq p > 1$ and we consider here Ω to be a bounded domain, one of the conclusions we can guarantee is that, for $p \in (1, N]$,

$$\frac{u}{\varphi^\tau} \in L^1(\Omega) \ , \quad \forall\, u \in W_0^{1,p}(\Omega) \ , \ \tau \in [0, 1]. \tag{A.2}$$

Now, we shall go through some embedding results which helps us relate norm in different Sobolev Spaces and $L^p(\Omega)$ spaces. These theorems are of most importance when treating compactness properties of the sets we are deeply interested. First, we begin with the Poincaré's Inequality, which readily implies the equivalence of the norms in $W_0^{k,p}(\Omega)$ and $W^{k,p}(\Omega)$.

Theorem A.0.3 (Poincaré's Inequality)**.** *Suppose $p \in [1, +\infty)$ and $u \in W_0^{1,p}(\Omega)$. Then we have the following estimate*

$$\left|u\right|_p \leq C \left|\nabla u\right|_p ,$$

with the constant $C > 0$ depending on p and Ω.

We recall that, given two Banach spaces X, Y, we say that X is *continuously* embedded in Y, denoted by $X \hookrightarrow Y$, if

1. $X \subset Y$;
2. The linear map $j : X \to Y$ given by $j(x) = x \in Y$, known as *canonical injection*, is a continuous operator. In other words, there exists a constant $C > 0$ such that $\|x\|_Y \leq C \|x\|_X$, for all $x \in X$.

Furthermore, we say that X is *compactly* embedded in Y if j is also a compact operator, meaning bounded sets of X are taken into relatively compact sets of Y. The most used definition of compact operator T, equivalent to the one above, is that every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X has a subsequence $(x_k)_{k\in\mathbb{N}}$ such that $(Tx_k)_{k\in\mathbb{N}}$ is convergent.

Let us now go through the most important embedding theorems we shall use.

Theorem A.0.4 (Sobolev Embedding Theorems)**.** *The following embedding are continuous*

- *If $1 \leq p < N$, $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.*

- *If* $p = N$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, *for* $q \in [p, +\infty)$.

- *If* $p > N$, $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.

In fact, Theorem A.0.4 is a particular, and more convenient, case of the following

Theorem A.0.5 (Sobolev-Morrey Embedding Theorems)**.** *Considering* $k \in \mathbb{N}$ *and* $p \geq 1$, *the following embedding are continuous*

- *If* $kp < N$, $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, *where* $q \in [1, p^*]$ *and* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

- *If* $kp = N$, $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, *for* $q \in [1, +\infty)$.

- *If* $p > N$, $W^{k,p}(\Omega) \hookrightarrow C^\tau(\overline{\Omega})$ *for some* $\tau \in (0,1)$. *In addition, if* Ω *has the strong Lipschitz properties, then we can actually affirm the continuous embedding* $W^{j+k,p}(\Omega) \hookrightarrow C^{j,\tau}(\overline{\Omega})$, $j \in \mathbb{N}$.

Proof. Its proof can be seen in [1, Theorem 5.4]. □

We notice that, given the Poincaré's Inequality, the norms in $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are equivalent. In addition, seen that the latter is contained in the former, Theorem A.0.4 readily implies that the same embedding results are still valid when we consider $W_0^{1,p}(\Omega)$. Furthermore, considering also the embedding $L^{p_1}(\Omega) \hookrightarrow L^{p_2}(\Omega)$ when $p_1 \geq p_2$ given by Theorem A.0.1, the following corollary can be extracted of Theorem A.0.4

Corollary A.0.5.1. *If* $p \in [1, N)$ *and* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$, *then we have the continuous embedding*

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \ , \quad \forall \, q \in [1, p^*].$$

In particular, $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, *for all* $q \in [1, 2^*]$, *meaning there exists a constant* $C > 0$ *such that*

$$|u|_q \leq C \, \|u\|_{H_0^1(\Omega)} \ , \quad \forall \, u \in H_0^1(\Omega).$$

At last, we give our final result, exploiting the cases when we obtain compact embbedings of the Sobolev spaces.

Theorem A.0.6 (Rellich-Kondrachov Embedding Theorems)**.** *Let* Ω *be an open bounded set in* $\mathbb{R}^N$ *with* C^1 *boundary. Then, the following embeddings are compact*

- *If* $1 \geq p < N$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, *for* $q \in [1, p^*)$, *where* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

- *If* $p = N$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, *for* $q \in [p, +\infty)$.

- *If $p > N$, $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$.*

Proof. See [2, Theorem 9.16] $\square$

The collection of all these results, which we shall refer as the Sobolev Embedding Theorems, together with some well known results, like the fact that reflexive spaces are weakly compact and that $L^p(\Omega)$ convergence implies a.e. convergence, let us conclude the following, which is an argument recurrent in this work,

Given a bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $W_0^{1,p}(\Omega)$, we obtain a subsequence (which we denote again by u_n) such that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } W_0^{1,p}(\Omega) \\ u_n \to u, & \text{in } L^q(\Omega)\ , q \in [1, p^*] \\ u_n \to u, & \text{a.e. in } \Omega. \end{cases}$$