Small violations of Bell inequalities for multipartite pure random states (Short Version)

Cristhiano Duarte,[1, 2, *] Raphael C. Drumond,[2, †] and Roberto I. Oliveira[3, ‡]

[1]*International Institute of Physics, Federal University of Rio Grande do Norte, 59078-970, P. O. Box 1613, Natal, Brazil*
[2]*Departamento de Matemática, Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, CP 702, CEP 30123-970, Belo Horizonte, MG, Brazil.*
[3]*Instituto Nacional de Matemática Pura e Aplicada-IMPA Estrada Dona Castorina, 110, Jardim Botânico 22460-320, Rio de Janeiro, RJ, Brazil*

(Dated: December 21, 2017)

I. INTRODUCTION

In his seminal paper [1], John Bell proved, in a simplified version of the bipartite scenario considered by Einstein, Podolsky and Rosen [2–4], that, under certain assumptions, there exist predictions of quantum theory incompatible with those of any physical theory which can be described by Local Hidden Variable (LHV) Models [5–7]. Roughly speaking, these local models are those in which outcomes of spacelike separated measurements are independent of each other when conditioned to the knowledge of a underlying hidden variable. Indeed, when a Bell scenario is fixed, it is always possible to obtain a (tight) linear inequality on joint probabilities, called a *Bell Inequality*, that holds for any theory that admits such a LHV description, and which might be violated otherwise. Since Bell's theorem is arguably one of most important results in quantum physics [8], with deep implications in our knowledge of the world [9–11] and growing interest in practical applications [7, 12, 13], it is natural asking about whether or not it is difficult to construct a real experimental setup within the quantum realm that violates one of these inequalities [14–18].

The present work answers one variant of that question. Using probabilistic techniques [5, 6, 19–23] , and generalizing previous results of the authors in [24], we are able to find an upper-bound to the typical behaviour of optimal violations, for any Bell scenario $\Gamma = (N, m, v)$, and to conclude in which extent the size of the local dimension d can influence [19] the probability to find N-partite d-dimensional quantum systems that violate any Bell inequality associated with this scenario. In fact, as we are going to prove, typical pure states *do not* produce large violations of Bell inequalities, apart from the fact that they are likely highly entangled [6, 7]. We remark that though that apparent paradoxical result, *i.e.* that of the higher dimensional multipartite states are, the more entangled they are though lesser Bell violations they show, our findings help to clarify the important difference (not always made clear) that there exist between the concept of entanglement and (Bell) non-locality.

The work is organized as follows. Section II is devoted to set up some notation and definitions that will be used throughout present work. In Section III we discuss the typical behaviour of high violations, whose result is formally expressed by Theorem 1. In Section IV we give all key ingredients necessary to prove it.

II. BELL INEQUALITIES

A. Basic Definitions

Throughout the work we will often consider as our starting point multipartite device-independent, or black-boxes, scenarios [7, 25], *i.e.* a general correlation scenario, denoted by

$$\Gamma := (N, m, v), \tag{1}$$

in which N black-boxes are distributed among N players, each of these boxes admitting m different inputs (always schematically represented within m different buttons at the top of each box), and such that for each input, among all v possible distinct outputs, only one outcome (frequently represented by lamps at the bottom of each box) is observed given that choice of input[26] (see Fig. 1).

* cduarte@iip.ufrn.br
† raphael@mat.ufmg.br
‡ rimfo@impa.br

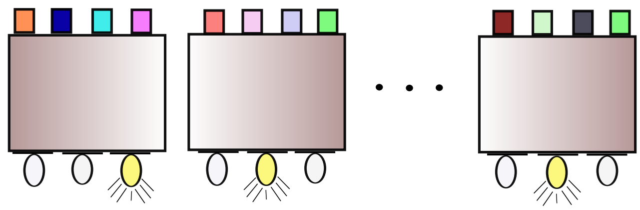

FIG. 1. Usual schematic representation of a Bell correlation scenario [7] $\Gamma = (N, m, v)$. Each box represents a device-independent apparatus whose internal details no one has access. Each button (input) above the boxes represents possible choices of measurements to be performed by the black-box. Different outcomes are represented by different bulbs above each box. When a particular outcome has been obtained the associated bulb glows indicating that an answer has been produced.

In addition, given that correlation scenario, since one does not have direct access to internal details of each box, the best description one has for Γ is through the correlations among inputs and outputs. It means that such experimental scenarios can be described then by a vector, usually called as *behaviour* [7, 27], $\vec{p} \in \mathbb{R}^{(vm)^N}$ whose components

$$p_{a_1,\dots,a_N,x_1,\dots,x_N} = P(a_1,\dots,a_N|x_1,\dots,x_N), \tag{2}$$

written in a neat manner, mean the joint probability for obtaining the outcome list $\{a_1,\dots,a_N\} \in [v] \times \dots \times [v]$ given that the inputs $\{x_1,\dots,x_N\} \in [m] \times \dots \times [m]$ have been chosen. Furthermore, any physically admissible behaviour $\vec{p} \in \mathbb{R}^{(mv)^N}$ must fulfil:

$$P(a_1,\dots,a_N|x_1,\dots,x_N) \in [0,1], \forall\, a_1,\dots,a_N, \forall\, x_1,\dots,x_N \tag{3}$$

$$\sum_{a_1,\dots,a_N} P(a_1,\dots,a_N|x_1,\dots,x_N) = 1,\ \forall\, x_1,\dots,x_m. \tag{4}$$

Let $\mathscr{B}_\Gamma$ to denote the set of *all admissible behaviours* satisfying the Eqs. (3) and (4) associated with the correlation scenario $\Gamma = (N, m, v)$.

The first physically motived additional constraint one could imagine is that of the local statistics for each subset of boxes does not depend on the choices of inputs of the other boxes outside this subset, *i.e.* that locally all marginals distributions are well-defined, and that there is no communication among any subset of boxes. In particular it would imply that for each part $i \in [N]$ the 1-box marginal distribution $\{P(a_i|x_i)\}$ is well-defined as well. More formally, a behaviour $\vec{p} \in \mathscr{B}_\Gamma$ is *non-signalling* when for all choices of subsets $\mathscr{I} \subset [N]$, say $\mathscr{I} = \{i_1,\dots,i_k\}$, one has

$$\sum_{\substack{a_i \\ i\in\mathscr{I}}} P(a_1,\dots,a_N|x_1,\dots,x_N) = P(a_{i_1},\dots,a_{i_k}|x_{i_1},\dots,x_{i_k}) = \sum_{\substack{a_i \\ i\in\mathscr{I}}} P(a_1,\dots,a_N|x'_1,\dots,x'_N), \forall\, a_{i_1},\dots,a_{i_k} \tag{5}$$

for all given inputs $\{x_1,\dots,x_N\}$ and $\{x'_1,\dots,x'_N\}$ whose intersection is the list $\{x_{i_1},\dots,x_{i_k}\}$. The set of all such behaviours being denoted by $\mathscr{B}_{\mathscr{N}\mathscr{S}}$, or $\mathscr{N}\mathscr{S}$ for short. In particular, if each one of the boxes at a scenario Γ is spacelike separated, the *non-signalling* constraints (5) assure that each part cannot use its own box to signal to other parts instantaneously, preventing therefore a direct conflict with relativity.

A more restrictive constraint appears when, given a scenario Γ, besides of being an admissible behaviour $\vec{p} \in \mathscr{B}_\Gamma$ one also requires that such a global correlation also possess a local explanation, *i.e.* that the possible lack of independence among parts:

$$P(a_1,\dots,a_N|x_1,\dots,x_N) \neq P(a_1|x_1) \times \dots \times P(a_N|x_N), \tag{6}$$

be only an effect of unknowing all possible sources of (classical) shared randomness [28, 29] amongst all parts involved at the scenario. Upon the knowledge of that hidden variable all local correlations would be independent of each other, and in that situation the global behaviour would be explained through the product of each probability distribution belonging only to each part[30]. More formally, given a scenario $\Gamma = (N, m, v)$ we say that a behaviour $\vec{p} \in \mathbb{R}^{(vm)^N}$ admits a *local hidden variable* (LHV) model, when there exist a probability space $\mathscr{P} = (\Omega, \Sigma, \mu)$, and response functions

$$\begin{aligned} p(a_i|x_i,\cdot) : \Omega &\longrightarrow [0,1] \\ \omega &\mapsto p(a_i|x_i,\omega) \end{aligned} \tag{7}$$

such that, for every input and output one has:

$$P(a_1,\dots,a_N|x_1,\dots,x_N) = \int_\Omega p(a_1|x_1,\omega)p(a_2|x_2,\omega)\dots p(a_N|x_N,\omega)\mu(d\omega). \tag{8}$$

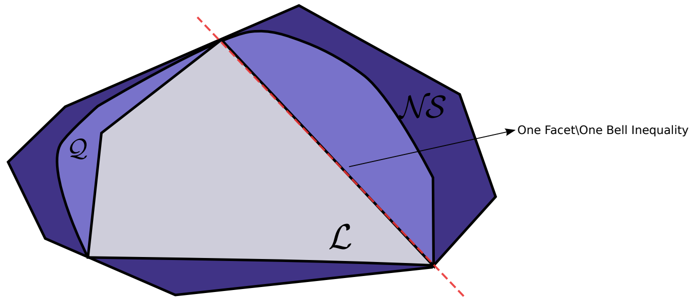

FIG. 2. Schematic drawing of geometrical aspects of the non-signalling, quantum and local sets. From their definitions it is clear that $\mathscr{L} \subset \mathscr{Q} \subset \mathscr{N}\mathscr{S}$ as suggest by the drawing above. In addition, firstly notice that while $\mathscr{N}\mathscr{S}$ and $\mathscr{L}$ are polytopes, the quantum set is not in general a polytope, even at the finite case. Secondly, as we have highlighted, every facet of the local polytope determines a valid Bell inequality [7] (colors online).

The set of all behaviours $\vec{p} \in \mathscr{B}_\Gamma$ which admit a LHV model is called *local set* and denoted by $\mathscr{B}_{\mathscr{L}}$, or $\mathscr{L}$ for short.

Since we are restricting ourselves to those cases in which the number of parts, the number of inputs as well as the number of outcomes are all of them finite, we can assume [7] that the underlying probability space is also finite and therefore rather than being just arbitrary convex sets, both $\mathscr{L}$ and $\mathscr{N}\mathscr{S}$ are indeed polytopes [31, 32]. In this geometrical framework the set of local behaviours is defined by a finite number of facets, or *Bell inequalities*:

$$\sum_{\substack{a_1,\dots,a_N\\ x_1,\dots,x_N}} T^f_{a_1,\dots,a_N|x_1,\dots,x_N} P(a_1,\dots,a_N|x_1,\dots,x_N) \leq \Delta, \tag{9}$$

where f in Eq. (9) above labels a particular choice of a facet [32]. We denote the set of all facets for the local polytope as $\mathscr{F}$. It is easily seen that any local behaviour satisfies all Bell inequalities, while on the other hand, if a behaviour $\vec{p}$ does not admit a LHV model, one must be able to find a facet f (described by one among many equivalent Bell inequalities) violated by such a non-local behaviour.

We should note that for our correlation scenario $\Gamma = (N,m,v)$, with N,m,v finite, the local polytope has number of vertices $a = v^{mN}$ and affine dimension $D = (mv)^N$. An upper bound for F_{max} its number of facets is:

$$F_{max} \leq 2\left[\frac{e(2v^{mN} - (mv)^N)}{(mv)^N - 2}\right]^{\frac{(mv)^N}{2}}. \tag{10}$$

A test of an inequality (9) can be designed by a quantum system through measurements of Bell Operators:

$$\mathfrak{B}^f := \sum_{\substack{a_1,\dots,a_N\\ x_1,\dots,x_N}} T^f_{a_1,\dots,a_N|x_1,\dots,x_N} \Pi^1_{a_1,x_1} \otimes \dots \otimes \Pi^N_{a_N,x_N}, \tag{11}$$

where for each part $k \in [N]$ each set $\{\Pi^k_{a_k,x_k}\}^v_{a_k=1}$ forms a POVM acting on a $d-$dimensional Hilbert space $\mathscr{H}_k$ associated with the $k-$th box. Therefore, given a pure quantum state $|\psi\rangle \in \bigotimes_{k=1}^N \mathscr{H}_k$, a choice $f \in \mathscr{F}$ of a facet, a collection A of POVM's, and a choice T of coefficients

$$T := \left\{T^f_{a_1,\dots,a_N|x_1,\dots,x_N}\right\}_{\substack{a_1,\dots,a_N\in[v]\\ x_1,\dots,x_N\in[m]}}, \tag{12}$$

with A and T both determining a valid Bell operator for f, a (possible) violation for such a Bell inequality is evaluated through the function:

$$Q(\psi,f,A,T) = Tr(\mathfrak{B}^f_{T,A}|\psi\rangle\langle\psi|). \tag{13}$$

Given a facet $f \in \mathscr{F}$, there are finitely many Bell inequalities associated with it, in fact as it will become clear later, throughout this work we are only interested on those Bell inequalities whose coefficients are uniformly bounded, therefore for that $f \in \mathscr{F}$ let $\mathscr{P}_f$ be the set of all possible choices local POVM's associated with all possible rewritings of Bell inequalities associated with f. Analogously, let $\mathscr{T}_f$ be all possible coefficients, uniformly bounded by M_Γ depending on Γ, associated with all those possible rewritings. Our strategy consists in, given $f \in \mathscr{F}$, optimizing the violation of Bell inequalities over $\mathscr{P}_f$ and $\mathscr{T}_f$, and then maximize it over all possible facets f (see Eq. (15) below).

Equation (13) arises from the usual definition of quantum correlations: given a scenario $\Gamma = (N, m, v)$, we say that a behaviour $\vec{p} \in \mathscr{B}_\Gamma$ belongs to set $\mathscr{Q}$ of *quantum correlations* when there exist a pure $N-$partite state $|\psi\rangle \in \otimes_{k=1}^N \mathscr{H}_k$, and POVM's $\{\Pi^k_{a_k,x_k}\}_{a_k=1}^v$ for each input x_k and each part $k \in [N]$ such that:

$$P(a_1,...,a_N|x_1,...,x_N) = Tr\left(\Pi^1_{a_1,x_1} \otimes ... \otimes \Pi^N_{a_N,x_N} |\psi\rangle\langle\psi|\right), \tag{14}$$

for all $a_1,...,a_N$ and for all $x_1,...,x_N$.

For our purposes, as it will become clear later on, it will be more suitable to consider Bell inequalities (9) whose coefficients are all uniformly bounded by a constant M_Γ. At the beginning it might seem very restrictive, since ultimately Bell inequalities are nothing but linear functionals separating the (closed) local set $\mathscr{L}$ from the other behaviours (see Fig. 2), and as such they are determined by those hyperplanes [33] whose intersection defines $\mathscr{L}$. It turns out that we can use the additional fact that elements lying in $\mathscr{B}_\Gamma$ obey the constraints expressed in Eqs. (3) and (4) to rewrite any Bell inequality in a neat way. Moreover, since Loubenets' approach is very general, any of these Bell inequalities is again a valid Bell inequality, and as such has already been considered in her optimization [6].

III. SMALL PROBABILITIES OF HIGH VIOLATIONS

Our main objective at the present work is to answer the following question:

given a random pure state $|\psi\rangle$ composed by N $d-$dimensional quantum systems, drawn accordingly to the uniform measure, what should one expect for its best possible violation when maximizing over all relevant Bell inequalities of a given scenario $\Gamma = (N, m, v)$?

It is already know [6, 7, 22] that if either d the local dimension of each quantum system, or the number N of parts, is sufficiently large then any quantum state is typically (according to the Haar measure) highly entangled. Might we expect then that typically there will also exist a high degree for the optimal violation when one optimizes over all possible Bell inequalities?

In our framework, optimal violations (if any) of (a huge class of) Bell inequalities exhibited by a quantum state $|\psi\rangle \in (\mathbb{C}^d)^{\otimes N}$ are given by the functional (see Eq. (13)):

$$V_{\text{opt}}(|\psi\rangle) := \sup_{f\in\mathscr{F}} \left(\sup_{\substack{A\in\mathscr{P}_f \\ T\in\mathscr{T}_f}} Q(\psi, f, A, T) \right), \tag{15}$$

where the supremum is taken over all possible facets $\mathscr{F}$ and all quantum implementations of (uniformly bounded) Bell inequalities associated with them.

If $|\psi\rangle$ is random variable, since V_{opt} is a function of ψ we can consider it as a random variable as well, then stated more formally what we would like to estimate is the distribution function of such variable, that is:

$$\mathbb{P}(V_{\text{opt}} > c), \tag{16}$$

for $c > 1$.

Our main result is the following

Theorem 1. *Given $N, d \geq 2$ integers. Let $|\psi\rangle \in \left(\mathbb{C}^d\right)^N$ be a unit vector distributed according to the uniform measure in the sphere S_{2d^N-1} of $\left(\mathbb{C}^d\right)^N$, then:*

$$\mathbb{P}(V_{\text{opt}} > c) \leq 4\left[\frac{e(2v^{mN} - (mv)^N)}{(mv)^N - 2}\right]^{\frac{(mv)^N}{2}} \times \left[\frac{2Nd^2(mv)^N}{\delta} + 1\right]^{mvNd^2} \times \left[\frac{M_\Gamma N(mv)^N}{\delta} + 1\right]^{(mv)^N} \times e^{-\left(\frac{2d^N(c-\delta-1)^2}{36\pi^2(2m-1)^{2N-2}}\right)}, \tag{17}$$

for any $\delta > 0$, $c > \delta + 1$.

This theorem allows us to answer the question posed at the beginning of this section negatively, *i.e. under some conditions* a typical state $|\psi\rangle$ composed by N d-dimensional quantum systems does not exhibit any significant degree of optimal violation for any Bell inequality. In fact, since

$$\mathbb{P}(V_{opt} > c) \leq 4\exp\left\{[(mv)^N](\frac{\log(2e)}{2} + \frac{mN\log(v)}{2} - \frac{\log[(mv)^N - 2]}{2}) + mvN^2d^2\log(mv) \right.$$
$$\left. + mvNd^2\log\left(\frac{4Nd^2}{\delta}\right) + (mv)^N\log\left(\frac{2N}{\delta}\right) + M_\Gamma N(mv)^N\log(mv) - \frac{(c-\delta-1)^2(2m-1)^2}{18\pi^2}\left[\frac{d}{(2m-1)^2}\right]^N\right\}, \tag{18}$$

if one assumes that d the local dimension of each subsystem satisfies

$$d > mv(2m-1)^2, \tag{19}$$

and that in addition the uniform bound M_Γ is not large enough, say, if

$$M_\Gamma = \mathcal{O}\left((mn)^N\right), \tag{20}$$

then the fourth term in brackets dominates all other terms, so we are left with:

$$\mathbb{P}(V_{\rm opt} > c) \to 0 \tag{21}$$

super-exponentially fast as $N \to \infty$.

Consequently:

if the local dimension d of a N-partite quantum system satisfies $d > mv(2m-1)^2$, with high probability, for large N, there is not any significant degree of optimal violation for any Bell inequality whose coefficients are not extremely large.

It shows that although on the one hand typically any N-partite pure state, with large N, is highly entangled, on the other hand their associated Bell violation is in general extraordinarily small.

IV. THE PROOF

A. Idea of the proof

Since our proof is going to be composed by many different pieces that we will glue together only at the very end, we dedicate this first subsection to present the main ideas and the whole strategy on which our argument relies. Roughly speaking, to obtain the bound

$$\mathbb{P}(V_{opt} > c) \leq 4 \underbrace{\left[\frac{e(2v^{mN} - (mv)^N)}{(mv)^N - 2}\right]^{\frac{(mv)^N}{2}}}_{(A)} \underbrace{\left[\frac{2Nd^2(mv)^N}{\delta} + 1\right]^{mvNd^2}}_{(B)} \underbrace{\left[\frac{M_\Gamma N(mv)^N}{\delta} + 1\right]^{(mv)^N}}_{(B')} \underbrace{\exp\left(-\frac{2d^N(c-\delta-1)^2}{36\pi^2(2m-1)^{2N-2}}\right)}_{(C)}, \tag{22}$$

on the probability of optimal violations of Bell inequalities, we use firstly some basic facts on probability plus an upper bound for the number of facets in a Bell scenario. This will be responsible for the factor (A) in Eq. 22. Secondly, we show how to approximate any set $X \subset [-1,1]^n$ contained in a n-dimensional hypercube by a finite set N_ε in such manner that for every point $x \in X$ there will exist another one $x' \in N_\varepsilon$ such that $\|x - x'\| \leq \varepsilon$. That ε-net technique will be used twice and is responsible for terms (B) and (B') in Eq. (22). The (C) term comes from Lévy's lemma [34] together with a couple of results on the smoothness of Q function.

---

[1] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).
[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[3] A. Aspect, P. Grangier, and G. Roger, Phys. Rev. Lett. **49**, 91 (1982).
[4] Stantford Encyclopedia of Phylosophy, "The Einstein-Podolsky-Rosen argument in quantum theory,".
[5] E. R. Loubenets, Journal of Mathematical Physics **53**, 022201 (2012), https://doi.org/10.1063/1.3681905.
[6] E. R. Loubenets, Foundations of Physics **47**, 1100 (2017).
[7] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).
[8] A. Cabello, *Pruebas algebraicas de imposibilidad de variables ocultas en Mecánica Cuántica*, Ph.D. thesis, Universidad Complutense de Madrid (1996).
[9] T. Vértesi and N. Brunner, Nature Communications **5**, 5297 (2014).
[10] J. B. D. Rosset and N. Gisin, J. Phys. A: Math. Theor. **47**, 424022 (2014).
[11] D. Avis, S. Moriyama, and M. Owari, IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences **E92.A**, 1254 (2009).
[12] J. I. de Vicente, Journal of Physics A: Mathematical and Theoretical **47**, 424017 (2014).
[13] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, Physical Review A **71**, 022101 (2005).
[14] A. Aspect, Physics Letters A **54**, 117 (1975).

[15] B. Hensen, H. Bernien, A. E. Dreau, A. Reiserer, N. Kalb, M. S. Blok, R. F. L. Ruitenberg, J.and Vermeulen, R. N. Schouten, C. Abellan, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau, and R. Hanson, Nature **526**, 682.
[16] B. Hensen, N. Kalb, M. S. Blok, A. E. Dréau, A. Reiserer, R. F. L. Vermeulen, R. N. Schouten, M. Markham, D. J. Twitchen, K. Goodenough, D. Elkouss, S. Wehner, T. H. Taminiau, and R. Hanson, Scientific Reports **6**, 30289 (2016).
[17] S. J. Freedman and J. F. Clauser, Phys. Rev. Lett. **28**, 938 (1972).
[18] A. Winter, Journal of Physics A: Mathematical and Theoretical **47**, 424031 (2014).
[19] C. E. González-Guillén, C. H. Jiménez, C. Palazuelos, and I. Villanueva, in *10th Conference on the Theory of Quantum Computation, Communication and Cryptography (TQC 2015)*, Leibniz International Proceedings in Informatics (LIPIcs), Vol. 44, edited by S. Beigi and R. Koenig (Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 2015) pp. 39–47.
[20] C. E. González-Guillén, C. Lancien, C. Palazuelos, and I. Villanueva, Annales Henri Poincaré **18**, 3793 (2017).
[21] C. Palazuelos and Z. Yin, Phys. Rev. A **92**, 052313 (2015).
[22] C. Palazuelos, ArXiv e-prints (2015), arXiv:1502.02175 [quant-ph].
[23] N. Alon and J. H. Spencer, *The Probabilistic Method*, Wiley-Interscience series in discrete mathematics and optimization.
[24] R. C. Drumond and R. I. Oliveira, Phys. Rev. A **86**, 012117 (2012).
[25] S. Pironio, V. Scarani, and T. Vidick, New Journal of Physics **18**, 100202 (2016).
[26] Notice that there is no loss of generality in supposing that every box has access to the same number m of inputs as well as to the same number v of outputs.This abstraction could be realized considering extra buttons that are never pushed and additional light bulbs that are never turned on.
[27] B. S. Tsirelson, *Hadronic Journal Supplement*, **8** (1993).
[28] J. I. de Vicente, Phys. Rev. A **95**, 012340 (2017).
[29] G. M. D'Ariano, P. L. Presti, and P. Perinotti, Journal of Physics A: Mathematical and General **38**, 5979 (2005).
[30] For more details, including historical remarks, we refer to references [5–7, 21, 22].
[31] A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*, Algorithms and Combinatorics (Springer, 2003).
[32] I. Pitowsky, Mathematical Programming **50**, 395 (1991).
[33] M. Laurent and V. F., *Lecture Notes on Semidefinite Optimization* (Amsterdam, NL, 2016).
[34] M. Ledoux, *The concentration of measure phenomenon*, 2nd ed. (American Mathematical Society, USA, 2001).