ENERGY DECAY FOR A WEAKLY DISSIPATIVE COUPLED WAVE SYSTEM IN $\mathbb{R}^n$

Abstract. The search for minimal damping mechanisms ensuring the uniform decay of the total energy to zero as time tends to infinity is a fundamental problem in the theory of dissipative evolution equations. A coupled system is said to be *weakly dissipative* when at least one of its equations does not contain a dissipative term. In this work, we investigate a coupled system of wave equations in $\mathbb{R}^n$ in which the second equation is undamped. We establish polynomial decay rates for the total energy associated with the linear system (1)–(2). Our analysis is carried out in the Fourier space and relies on the construction of suitable Fourier energy functionals, the monotonicity of both local and total energies, and the integrability properties of singularities arising near the origin in the frequency domain. The techniques developed here provide a unified framework for studying the long-time behavior of a broad class of weakly dissipative linear coupled systems posed in $\mathbb{R}^n$.

We consider the following coupled system of wave equations:

$$\begin{aligned} u_{tt}(t,x) - \Delta u(t,x) + \beta u_t(t,x) + \lambda u(t,x) - \kappa v(t,x) &= 0, \\ v_{tt}(t,x) - \Delta v(t,x) + \mu v(t,x) - \kappa u(t,x) &= 0, \end{aligned} \tag{1}$$

for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, where β, λ, μ, and κ are real constants. The initial data are given by

$$\begin{aligned} u(0,x) = u_0(x) \quad &\text{and} \quad u_t(0,x) = u_1(x), \\ v(0,x) = v_0(x) \quad &\text{and} \quad v_t(0,x) = v_1(x), \end{aligned} \tag{2}$$

for all $x \in \mathbb{R}^n$. This model describes the dynamics of two elastic membranes coupled through an elastic restoring force that tends to bring one membrane toward the other.

To derive decay estimates for the total energy, we analyze system (1) in the Fourier space. Applying the Fourier transform with respect to the spatial variable x, we obtain

$$\hat{u}_{tt}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) + \beta \hat{u}_t(t,\xi) + \lambda \hat{u}(t,\xi) - \kappa \hat{v}(t,\xi) = 0, \tag{3}$$

$$\hat{v}_{tt}(t,\xi) + |\xi|^2 \hat{v}(t,\xi) + \mu \hat{v}(t,\xi) - \kappa \hat{u}(t,\xi) = 0, \tag{4}$$

for all $\xi \in \mathbb{R}^n$ and $t > 0$. The initial data in the Fourier space are given by

$$\begin{aligned} \hat{u}(0,\xi) = \hat{u}_0(\xi) \quad &\text{and} \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi), \\ \hat{v}(0,\xi) = \hat{v}_0(\xi) \quad &\text{and} \quad \hat{v}_t(0,\xi) = \hat{v}_1(\xi), \end{aligned}$$

for all $\xi \in \mathbb{R}^n$.

Multiplying equation (3) by $\overline{\hat{u}_t}$ and equation (4) by $\overline{\hat{v}_t}$, taking the real parts, and adding the resulting identities, we obtain

$$\frac{d}{dt}(\mathcal{E}(t,\xi)) + \beta |\hat{u}_t(t,\xi)|^2 = 0, \quad \xi \in \mathbb{R}^n \text{ and } t > 0,$$

where the Fourier energy density is defined by

$$\mathcal{E}(t,\xi) = \frac{1}{2}\left\{ |\hat{u}_t(t,\xi)|^2 + \left(|\xi|^2 + \lambda\right)|\hat{u}(t,\xi)|^2 + |\hat{v}_t(t,\xi)|^2 + \left(|\xi|^2 + \lambda\right)|\hat{v}(t,\xi)|^2 - \kappa Re\left[\hat{u}(t,\xi)\,\overline{\hat{v}}(t,\xi)\right]\right\}.$$

The total energy in the Fourier space is then defined by

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^n} \Big(|\hat{u}_t(t,\xi)|^2 + \big(|\xi|^2+\lambda\big)\,|\hat{u}(t,\xi)|^2 + |\hat{v}_t(t,\xi)|^2 + \big(|\xi|^2+\lambda\big)\,|\hat{v}(t,\xi)|^2 - \kappa Re\left[\hat{u}(t,\xi)\,\overline{\hat{v}}(t,\xi)\right] \Big)\, d\xi,$$

for any $t \geq 0$.

1. Main Results

Our main result establishes polynomial decay rates for the total energy associated with solutions to system (1)–(2).

Theorem 1.1. *Let $n \geq 1$, and let β, λ, μ, and κ be constants satisfying suitable assumptions. If $[u_0, u_1, v_0, v_1] \in H^{1+\frac{1}{\alpha}}(\mathbb{R}^n) \times H^{\frac{1}{\alpha}}(\mathbb{R}^n) \times H^{1+\frac{1}{\alpha}}(\mathbb{R}^n) \times H^{\frac{1}{\alpha}}(\mathbb{R}^n)$, there exists a positive constant $C_\alpha > 0$ depending on α, such that the total energy of the system* (1)-(2)*, in Fourier space, satisfies*

$$E(t) \leq C_\alpha \big\{ \|u_0\|^2_{H^{1+\frac{1}{\alpha}}} + \|u_1\|^2_{H^{\frac{1}{\alpha}}} + \|v_0\|^2_{H^{1+\frac{1}{\alpha}}} + \|v_1\|^2_{H^{\frac{1}{\alpha}}} \big\}\, t^{-1/\alpha},$$

for every $t \geq T_0$ where T_0 is a constant depending on the initial data.

As an immediate consequence of Theorem 1.1, we obtain the following corollary.

Corollary 1.2. *Under the assumptions of Theorem 1.1 then there exists a constant $C_\alpha > 0$ depending on α, such that the corresponding solution $(u(t,x), v(t,x))$ of the system* (1)-(2) *satisfies:*

$$\|u(t)\|^2_{H^1} + \|u_t(t)\|^2_{L^2} + \|v(t)\|^2_{H^1} + \|v_t(t)\|^2_{L^2} \leq C_\alpha \big\{ \|u_0\|^2_{H^{1+\frac{1}{\alpha}}} + \|u_1\|^2_{H^{\frac{1}{\alpha}}} + \|v_0\|^2_{H^{1+\frac{1}{\alpha}}} + \|v_1\|^2_{H^{\frac{1}{\alpha}}} \big\}\, t^{-1/\alpha},$$

for all $t \geq T_0$ where T_0 is a constant depending on the initial data.

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