Order statistics

Tiago M. Magalhães

Department of Statistics - ICE-UFJF

Juiz de Fora, November 06, 2026

Outline

1. Order statistics
2. Distribution of order statistics
3. Distribution of the sample range
4. Bibliography

Outline

1. Order statistics
2. Distribution of order statistics
3. Distribution of the sample range
4. Bibliography

Order statistics

Let $X_1, X_2, \ldots, X_n$ be IID RVs, with distribution $\mathcal{D}$, probability density function (PDF) f, and cumulative distribution function (CDF) F. Define:

$$X_{(1)} = \text{the smallest value among } X_1, X_2, \ldots, X_n;$$

$$X_{(2)} = \text{the second smallest value among } X_1, X_2, \ldots, X_n;$$

$$\vdots$$

$$X_{(j)} = \text{the } j\text{th smallest value among } X_1, X_2, \ldots, X_n;$$

$$\vdots$$

$$X_{(n)} = \text{the largest value among } X_1, X_2, \ldots, X_n.$$

Order statistics

Let $X_1, X_2, \ldots, X_n$ be IID RVs, with distribution $\mathcal{D}$, probability density function (PDF) f, and cumulative distribution function (CDF) F. Define:

$$X_{(1)} = \text{the smallest value among } X_1, X_2, \ldots, X_n;$$

$$X_{(2)} = \text{the second smallest value among } X_1, X_2, \ldots, X_n;$$

$$\vdots$$

$$X_{(j)} = \text{the } j\text{th smallest value among } X_1, X_2, \ldots, X_n;$$

$$\vdots$$

$$X_{(n)} = \text{the largest value among } X_1, X_2, \ldots, X_n.$$

Order statistics

The ordered values $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are called the **order statistics** corresponding to the RVs $X_1, X_2, \ldots, X_n$.

In other words, $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the ordered values of $X_1, X_2, \ldots, X_n$.

Order statistics

The ordered values $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are called the **order statistics** corresponding to the RVs $X_1, X_2, \ldots, X_n$.

In other words, $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the ordered values of $X_1, X_2, \ldots, X_n$.

Order statistics

Remark

The extreme values $X_{(1)}$ and $X_{(n)}$ are often the most interesting, as they are, respectively, the **minimum** and the **maximum**, that is, $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$.

Order statistics

Remark

The extreme values $X_{(1)}$ and $X_{(n)}$ are often the most interesting, as they are, respectively, the **minimum** and the **maximum**, that is, $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$.

Examples

Example 1. If $X_1, \ldots, X_5$ are IID exponential RVs with parameter λ, calculate:

(a) $\mathbb{P}\left(\min\left\{X_1, \ldots, X_5\right\} \leq a\right)$;

(b) $\mathbb{P}\left(\max\left\{X_1, \ldots, X_5\right\} \leq a\right)$.

Recalling that, if $X \sim \text{Exp}(\lambda)$, then

$$F_X(x) = 1 - \exp\left\{-\lambda x\right\}, \; S_X(x) = 1 - F_X(x) = \exp\left\{-\lambda x\right\}. \qquad (1)$$

Examples

Example 1. If $X_1, \ldots, X_5$ are IID exponential RVs with parameter λ, calculate:

(a) $\mathbb{P}\left(\min\left\{X_1, \ldots, X_5\right\} \leq a\right)$;

(b) $\mathbb{P}\left(\max\left\{X_1, \ldots, X_5\right\} \leq a\right)$.

Recalling that, if $X \sim \text{Exp}(\lambda)$, then

$$F_X(x) = 1 - \exp\left\{-\lambda x\right\}, \; S_X(x) = 1 - F_X(x) = \exp\left\{-\lambda x\right\}. \qquad (1)$$

Outline

1 Order statistics

2 Distribution of order statistics

3 Distribution of the sample range

4 Bibliography

Order statistics

The joint PDF of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is given by

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! \times f_{X_1}(x_1) \times \cdots \times f_{X_n}(x_n)\, \mathbb{I}_{\mathbb{R}}(x_1 < \cdots < x_n).$$

Order statistics

And the marginal PDF of the jth order statistic $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(n-j)!(j-1)!} [F(x)]^{j-1} [1 - F(x)]^{n-j} f(x). \qquad (2)$$

Order statistics

For $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$, the PDF in (2) can be written, respectively, as follows:

$$f_{X_{(1)}}(x) = n\,[1 - F(x)]^{n-1}\, f(x), \quad f_{X_{(n)}}(x) = n\,[F(x)]^{n-1}\, f(x).$$

Order statistics

For $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$, the PDF in (2) can be written, respectively, as follows:

$$f_{X_{(1)}}(x) = n\left[1 - F(x)\right]^{n-1} f(x), \quad f_{X_{(n)}}(x) = n\left[F(x)\right]^{n-1} f(x).$$

Examples

Example 2. We say that an RV follows a truncated exponential distribution with parameters $\lambda > 0$ and $\theta > 0$,

Examples

Example 2. We say that an RV follows a truncated exponential distribution with parameters $\lambda > 0$ and $\theta > 0$, $X \sim \text{TE}(\lambda, \theta)$,

Examples

Example 2. We say that an RV follows a truncated exponential distribution with parameters $\lambda > 0$ and $\theta > 0$, $X \sim \text{TE}(\lambda, \theta)$, if its PDF is of the form

$$f(x) = \lambda e^{-\lambda(x-\theta)} \mathbb{I}_{(\theta,\infty)}(x).$$

Examples

Example 2. We say that an RV follows a truncated exponential distribution with parameters $\lambda > 0$ and $\theta > 0$, $X \sim \text{TE}(\lambda, \theta)$, if its PDF is of the form

$$f(x) = \lambda e^{-\lambda(x-\theta)} \mathbb{I}_{(\theta,\infty)}(x).$$

Let $X_1, X_2, \ldots, X_n$ be a random sample from $X \sim \text{TE}(1, \theta)$. Find the PDF of $Y = \min\{X_1, X_2, \ldots, X_n\}$.

Examples

Example 2. We say that an RV follows a truncated exponential distribution with parameters $\lambda > 0$ and $\theta > 0$, $X \sim \text{TE}(\lambda, \theta)$, if its PDF is of the form

$$f(x) = \lambda e^{-\lambda(x-\theta)} \mathbb{I}_{(\theta,\infty)}(x).$$

Let $X_1, X_2, \ldots, X_n$ be a random sample from $X \sim \text{TE}(1, \theta)$. Find the PDF of $Y = \min\{X_1, X_2, \ldots, X_n\}$.

Examples

First, it is important to know that, if $X \sim \text{TE}(\lambda, \theta)$, then

$$F_X(x) = 1 - \exp\{-\lambda(x-\theta)\}, \ \mathbb{E}(X) = \frac{1}{\lambda} + \theta, \ \text{Var}(X) = \frac{1}{\lambda^2}. \qquad (3)$$

Examples

First, it is important to know that, if $X \sim \text{TE}(\lambda, \theta)$, then

$$F_X(x) = 1 - \exp\{-\lambda(x-\theta)\}, \ \mathbb{E}(X) = \frac{1}{\lambda} + \theta, \ \text{Var}(X) = \frac{1}{\lambda^2}. \qquad (3)$$

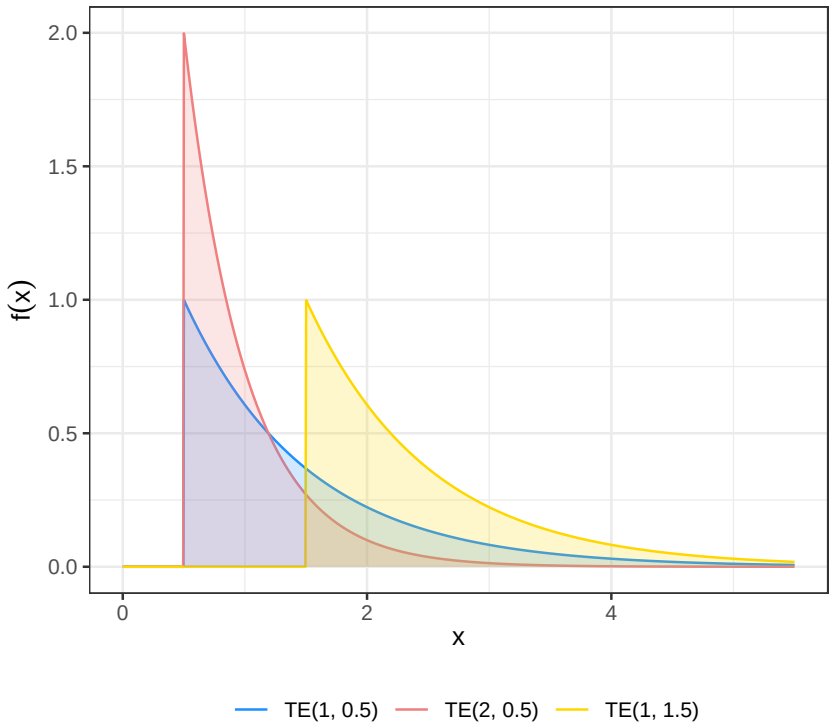
Note that $X = Z + \theta$, where $Z \sim \text{Exp}(\lambda)$.

Examples

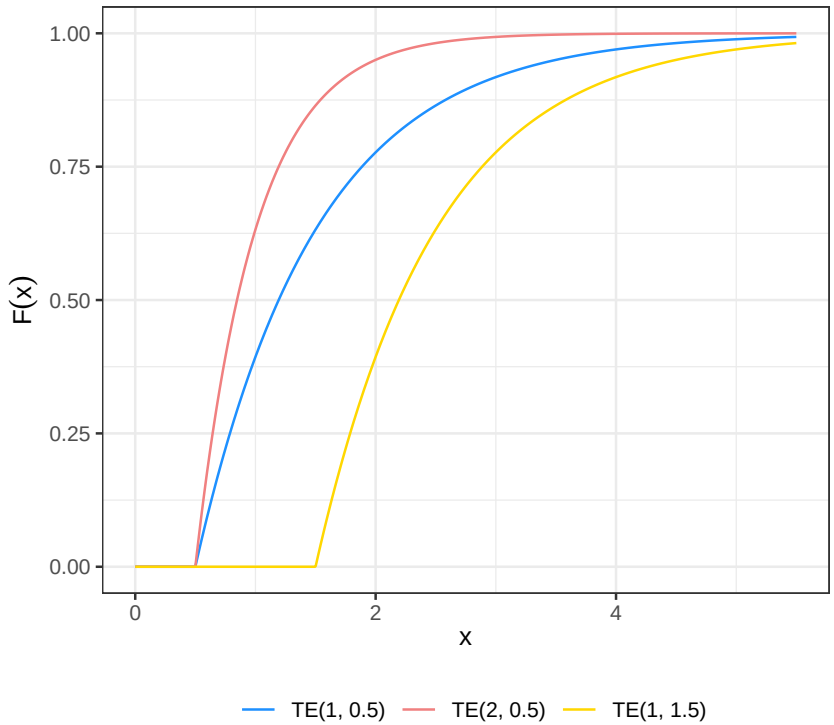
First, it is important to know that, if $X \sim \text{TE}(\lambda, \theta)$, then

$$F_X(x) = 1 - \exp\{-\lambda(x-\theta)\}, \ \mathbb{E}(X) = \frac{1}{\lambda} + \theta, \ \text{Var}(X) = \frac{1}{\lambda^2}. \qquad (3)$$

Note that $X = Z + \theta$, where $Z \sim \text{Exp}(\lambda)$.



(a) PDF.



(b) CDF.

Figure 1: Truncated exponential distribution for different values of λ and θ.

Outline

1 Order statistics

2 Distribution of order statistics

3 Distribution of the sample range

4 Bibliography

Sample range

Again, let $X_1, X_2, \ldots, X_n$ be IID RVs, with distribution $\mathcal{D}$, PDF f, and CDF F. The RV

$$R = X_{(n)} - X_{(1)}$$

is called the **sample range** of the observed RVs.

Sample range

Again, let $X_1, X_2, \ldots, X_n$ be IID RVs, with distribution $\mathcal{D}$, PDF f, and CDF F. The RV

$$R = X_{(n)} - X_{(1)}$$

is called the **sample range** of the observed RVs.

Sample range

In general, the distribution of R can be obtained from the definition of the CDF, as follows:

$$G(r) = \mathbb{P}(R \leq r) = n \int_{-\infty}^{+\infty} [F(x+r) - F(x)]^{n-1} f(x)\, dx. \qquad (4)$$

Sample range

In general, the distribution of R can be obtained from the definition of the CDF, as follows:

$$G(r) = \mathbb{P}(R \leq r) = n \int_{-\infty}^{+\infty} [F(x+r) - F(x)]^{n-1} f(x)\, dx. \qquad (4)$$

Expression (4) has an explicit form in only a few cases.

Sample range

In general, the distribution of R can be obtained from the definition of the CDF, as follows:

$$G(r) = \mathbb{P}(R \leq r) = n \int_{-\infty}^{+\infty} [F(x+r) - F(x)]^{n-1} f(x)\, dx. \qquad (4)$$

Expression (4) has an explicit form in only a few cases.

Sample range

Remark

The sample range is widely used in Statistical Quality Control.

Examples

Example 3. If $X_1, X_2, \ldots, X_n$ are IID Uniform$(0, 1)$ RVs, find the distribution of:

$$R = X_{(n)} - X_{(1)}.$$

Outline

1 Order statistics

2 Distribution of order statistics

3 Distribution of the sample range

4 Bibliography

Bibliography

Meyer, P. L. (2017). *Probabilidade: aplicações à Estatística*. Livros Técnicos e Científicos Editora Ltda, Rio de Janeiro, 2nd edition.

Ross, S. (2019). *A first course in probability*. Pearson Education, Malaysia, 10th edn. global edition.

Thank you!

✉ tiago.magalhaes@ufjf.br

ufjf.br/tiago_magalhaes

Department of Statistics, Room 319