

Jacobian method

Tiago M. Magalhães

Department of Statistics - ICE-UFJF

Juiz de Fora, October 02, 2026



Outline

- 1 Introduction
- 2 Discrete case
- 3 Jacobian method
- 4 Non-one-to-one functions
- 5 Bibliography



Outline

- 1 Introduction
- 2 Discrete case
- 3 Jacobian method
- 4 Non-one-to-one functions
- 5 Bibliography



Introduction

Motivation

Given the joint distribution of $(X_1, X_2)^\top$, what is the distribution of $Z = H(X_1, X_2)$? Or what is the joint distribution of $Y_1 = H_1(X_1, X_2)$ and $Y_2 = H_2(X_1, X_2)$?



Introduction

Motivation

Given the joint distribution of $(X_1, X_2)^\top$, what is the distribution of $Z = H(X_1, X_2)$? Or what is the joint distribution of $Y_1 = H_1(X_1, X_2)$ and $Y_2 = H_2(X_1, X_2)$?



Outline

- 1 Introduction
- 2 Discrete case**
- 3 Jacobian method
- 4 Non-one-to-one functions
- 5 Bibliography



Discrete case

If $(X, Y)^\top$ is a discrete random vector, finding the probability distribution of $Z = H(X, Y)$ is a relatively simple task.



Example

What is the probability distribution of $Z = \min(X, Y)$, where $(X, Y)^T$ has probability distribution given by:

Table 1: Joint probability mass function of X and Y .

Y	X					
	0	1	2	3	4	5
0	0.00	0.01	0.03	0.05	0.07	0.09
1	0.01	0.02	0.04	0.05	0.06	0.08
2	0.01	0.03	0.05	0.05	0.05	0.06
3	0.01	0.02	0.04	0.06	0.06	0.05



Example

What is the probability distribution of $Z = \min(X, Y)$, where $(X, Y)^T$ has probability distribution given by:

Table 1: Joint probability mass function of X and Y .

Y	X					
	0	1	2	3	4	5
0	0.00	0.01	0.03	0.05	0.07	0.09
1	0.01	0.02	0.04	0.05	0.06	0.08
2	0.01	0.03	0.05	0.05	0.05	0.06
3	0.01	0.02	0.04	0.06	0.06	0.05



Example

Note that the support of Z is $\mathcal{Z} = \{0, 1, 2, 3\}$, obtained as follows:

Z	(X, Y)									
0	(0,0)	(0,1)	(0,2)	(0,3)	(1,0)	(2,0)	(3,0)	(4,0)	(5,0)	
1	(1,1)	(1,2)	(1,3)	(2,1)	(3,1)	(4,1)	(5,1)			
2	(2,2)	(2,3)	(3,2)	(4,2)	(5,2)					
3	(3,3)	(4,3)	(5,3)							

Example

Note that the support of Z is $\mathcal{Z} = \{0, 1, 2, 3\}$, obtained as follows:

Z	(X, Y)									
0	(0,0)	(0,1)	(0,2)	(0,3)	(1,0)	(2,0)	(3,0)	(4,0)	(5,0)	
1	(1,1)	(1,2)	(1,3)	(2,1)	(3,1)	(4,1)	(5,1)			
2	(2,2)	(2,3)	(3,2)	(4,2)	(5,2)					
3	(3,3)	(4,3)	(5,3)							

Example

Using the joint probability mass function,

$$\begin{aligned}\mathbb{P}(Z = 0) &= \mathbb{P}(X = 0, Y = 0) + \mathbb{P}(X = 0, Y = 1) + \mathbb{P}(X = 0, Y = 2) \\ &\quad + \mathbb{P}(X = 0, Y = 3) + \mathbb{P}(X = 1, Y = 0) + \mathbb{P}(X = 2, Y = 0) \\ &\quad + \mathbb{P}(X = 3, Y = 0) + \mathbb{P}(X = 4, Y = 0) + \mathbb{P}(X = 5, Y = 0); \\ \mathbb{P}(Z = 1) &= \mathbb{P}(X = 1, Y = 1) + \mathbb{P}(X = 1, Y = 2) + \mathbb{P}(X = 1, Y = 3) \\ &\quad + \mathbb{P}(X = 2, Y = 1) + \mathbb{P}(X = 3, Y = 1) + \mathbb{P}(X = 4, Y = 1) \\ &\quad + \mathbb{P}(X = 5, Y = 1);\end{aligned}$$



Example

$$\begin{aligned}\mathbb{P}(Z = 2) &= \mathbb{P}(X = 2, Y = 2) + \mathbb{P}(X = 2, Y = 3) + \mathbb{P}(X = 3, Y = 2) \\ &\quad + \mathbb{P}(X = 4, Y = 2) + \mathbb{P}(X = 5, Y = 2); \\ \mathbb{P}(Z = 3) &= \mathbb{P}(X = 3, Y = 3) + \mathbb{P}(X = 4, Y = 3) + \mathbb{P}(X = 5, Y = 3).\end{aligned}$$

Example

The probability distribution of the RV $Z = \min(X, Y)$ is given by:

Table 2: Probability mass function of $Z = \min(X, Y)$.

Z	0	1	2	3
$P(Z)$	0.28	0.30	0.25	0.17

Example

The probability distribution of the RV $Z = \min(X, Y)$ is given by:

Table 2: Probability mass function of $Z = \min(X, Y)$.

Z	0	1	2	3
$P(Z)$	0.28	0.30	0.25	0.17

Outline

- 1 Introduction
- 2 Discrete case
- 3 Jacobian method**
- 4 Non-one-to-one functions
- 5 Bibliography



Continuous case

If $(X, Y)^T$ is a continuous bivariate RV and if $Z = H(X, Y)$ is a continuous function of (X, Y) , then Z will be a (univariate) continuous RV, and the procedure for finding its probability density function (PDF) is somewhat more complicated.



Continuous case

In order to find the PDF of Z , we will need the **Jacobian method**.



Jacobian method

Suppose that $(X_1, X_2)^\top$ is a continuous bivariate RV with joint PDF f . Let $Y_1 = H_1(X_1, X_2)$ and $Y_2 = H_2(X_1, X_2)$, and assume that the functions H_1 and H_2 satisfy the following conditions



Jacobian method

Suppose that $(X_1, X_2)^\top$ is a continuous bivariate RV with joint PDF f . Let $Y_1 = H_1(X_1, X_2)$ and $Y_2 = H_2(X_1, X_2)$, and assume that the functions H_1 and H_2 satisfy the following conditions

- The equations $y_1 = H_1(x_1, x_2)$ and $y_2 = H_2(x_1, x_2)$ can be **uniquely** solved for x_1 and x_2 , in terms of y_1 and y_2 , that is, $x_1 = H_1^{-1}(y_1, y_2)$ and $x_2 = H_2^{-1}(y_1, y_2)$.



Jacobian method

Suppose that $(X_1, X_2)^\top$ is a continuous bivariate RV with joint PDF f . Let $Y_1 = H_1(X_1, X_2)$ and $Y_2 = H_2(X_1, X_2)$, and assume that the functions H_1 and H_2 satisfy the following conditions

- The equations $y_1 = H_1(x_1, x_2)$ and $y_2 = H_2(x_1, x_2)$ can be **uniquely** solved for x_1 and x_2 , in terms of y_1 and y_2 , that is, $x_1 = H_1^{-1}(y_1, y_2)$ and $x_2 = H_2^{-1}(y_1, y_2)$.



Jacobian method

- The partial derivatives

$$\frac{\partial H_1(x_1, x_2)}{\partial x_1}, \frac{\partial H_1(x_1, x_2)}{\partial x_2}, \frac{\partial H_2(x_1, x_2)}{\partial x_1}, \frac{\partial H_2(x_1, x_2)}{\partial x_2}$$

exist, are continuous, and



Jacobian method

such that:

$$\mathcal{J} = J(x_1, x_2) = \begin{vmatrix} \frac{\partial H_1(x_1, x_2)}{\partial x_1} & \frac{\partial H_1(x_1, x_2)}{\partial x_2} \\ \frac{\partial H_2(x_1, x_2)}{\partial x_1} & \frac{\partial H_2(x_1, x_2)}{\partial x_2} \end{vmatrix} \neq 0, \quad (1)$$

at all points (x_1, x_2) . The determinant in (1) is called the **Jacobian** of the transformation $(x_1, x_2) \rightarrow (y_1, y_2)$.



Jacobian method

such that:

$$\mathcal{J} = J(x_1, x_2) = \begin{vmatrix} \frac{\partial H_1(x_1, x_2)}{\partial x_1} & \frac{\partial H_1(x_1, x_2)}{\partial x_2} \\ \frac{\partial H_2(x_1, x_2)}{\partial x_1} & \frac{\partial H_2(x_1, x_2)}{\partial x_2} \end{vmatrix} \neq 0, \quad (1)$$

at all points (x_1, x_2) . The determinant in (1) is called the **Jacobian** of the transformation $(x_1, x_2) \rightarrow (y_1, y_2)$.



Jacobian method

Under these circumstances, the joint PDF of $(Y_1, Y_2)^\top$, that is, $g(y_1, y_2)$ is given by the following expression

$$g(y_1, y_2) = f \left[H_1^{-1}(y_1, y_2), H_2^{-1}(y_1, y_2) \right] |\mathcal{J}|^{-1}, \quad (2)$$

where $f \left[H_1^{-1}(y_1, y_2), H_2^{-1}(y_1, y_2) \right]$ is the joint PDF of $(X_1, X_2)^\top$ evaluated at $x_1 = H_1^{-1}(y_1, y_2)$, $x_2 = H_2^{-1}(y_1, y_2)$



Jacobian method

Under these circumstances, the joint PDF of $(Y_1, Y_2)^\top$, that is, $g(y_1, y_2)$ is given by the following expression

$$g(y_1, y_2) = f \left[H_1^{-1}(y_1, y_2), H_2^{-1}(y_1, y_2) \right] |\mathcal{J}|^{-1}, \quad (2)$$

where $f \left[H_1^{-1}(y_1, y_2), H_2^{-1}(y_1, y_2) \right]$ is the joint PDF of $(X_1, X_2)^\top$ evaluated at $x_1 = H_1^{-1}(y_1, y_2)$, $x_2 = H_2^{-1}(y_1, y_2)$ and $\mathcal{J}$ is the Jacobian given in (1), expressed as a function of y_1 and y_2 .



Jacobian method

Under these circumstances, the joint PDF of $(Y_1, Y_2)^\top$, that is, $g(y_1, y_2)$ is given by the following expression

$$g(y_1, y_2) = f \left[H_1^{-1}(y_1, y_2), H_2^{-1}(y_1, y_2) \right] |\mathcal{J}|^{-1}, \quad (2)$$

where $f \left[H_1^{-1}(y_1, y_2), H_2^{-1}(y_1, y_2) \right]$ is the joint PDF of $(X_1, X_2)^\top$ evaluated at $x_1 = H_1^{-1}(y_1, y_2)$, $x_2 = H_2^{-1}(y_1, y_2)$ and $\mathcal{J}$ is the Jacobian given in (1), expressed as a function of y_1 and y_2 .



Examples

Example 1. If X and Y are two independent RVs with Gamma distributions with shape and scale parameters (α, λ) and (β, λ) , respectively, find the joint PDF of

$$U = X + Y \text{ and } V = \frac{X}{X + Y}.$$

Jacobian method

We may be interested only in a function of two random variables X_1 and X_2 , that is, we are interested in:

$$Y = H(X_1, X_2).$$



Jacobian method

We may be interested only in a function of two random variables X_1 and X_2 , that is, we are interested in:

$$Y = H(X_1, X_2).$$

However, to apply the Jacobian method in this case, we introduce a second function, called an auxiliary function.



Jacobian method

We may be interested only in a function of two random variables X_1 and X_2 , that is, we are interested in:

$$Y = H(X_1, X_2).$$

However, to apply the Jacobian method in this case, we introduce a second function, called an auxiliary function.



Jacobian method

Let $Y_1 = H_1(X_1, X_2)$ be the function of interest and $Y_2 = H_2(X_1, X_2)$ the auxiliary function; we apply the Jacobian method to find the joint PDF of Y_1 and Y_2 .



Jacobian method

Knowing $g(y_1, y_2)$, we can then obtain the marginal PDF of Y_1 , $g_{Y_1}(y_1)$, by integrating $g(y_1, y_2)$ with respect to y_2 , that is,

$$g_{Y_1}(y_1) = \int_{-\infty}^{+\infty} g(y_1, y_2) dy_2.$$



Examples

Example 2. Let X_1 and X_2 be two independent random variables with chi-square(m) and chi-square(n) distributions, respectively. Let

$$Y = \frac{X_1/m}{X_2/n}.$$

What is the distribution of Y ?



Examples

Example 2. Let X_1 and X_2 be two independent random variables with chi-square(m) and chi-square(n) distributions, respectively. Let

$$Y = \frac{X_1/m}{X_2/n}.$$

What is the distribution of Y ?



Outline

- 1 Introduction
- 2 Discrete case
- 3 Jacobian method
- 4 Non-one-to-one functions**
- 5 Bibliography



Jacobian method

When the function $Y = H(X_1, X_2)$ is not one-to-one, the Jacobian method, described in (2), cannot be used directly.



Jacobian method

First, the support of (X_1, X_2) must be subdivided into regions where $H(X_1, X_2)$ is one-to-one.

After that, the method can be applied to each region, and the PDF of Y is obtained by summing the PDFs obtained over all regions.



Jacobian method

First, the support of (X_1, X_2) must be subdivided into regions where $H(X_1, X_2)$ is one-to-one.

After that, the method can be applied to each region, and the PDF of Y is obtained by summing the PDFs obtained over all regions.



Jacobian method

That is, the joint PDF of $(Y_1, Y_2)^\top$ will be given by the expression

$$g(y_1, y_2) = \sum_{i=1}^n f \left[H_{1i}^{-1}(y_1, y_2), H_{2i}^{-1}(y_1, y_2) \right] |\mathcal{J}_i|^{-1}, \quad (3)$$

where $\mathcal{J}_i$ is the Jacobian corresponding to the i th region, expressed as a function of y_1 and y_2 , and n is the number of subdivisions.



Jacobian method

That is, the joint PDF of $(Y_1, Y_2)^\top$ will be given by the expression

$$g(y_1, y_2) = \sum_{i=1}^n f \left[H_{1i}^{-1}(y_1, y_2), H_{2i}^{-1}(y_1, y_2) \right] |\mathcal{J}_i|^{-1}, \quad (3)$$

where $\mathcal{J}_i$ is the Jacobian corresponding to the i th region, expressed as a function of y_1 and y_2 , and n is the number of subdivisions.



Outline

- 1 Introduction
- 2 Discrete case
- 3 Jacobian method
- 4 Non-one-to-one functions
- 5 Bibliography



Bibliography

- Meyer, P. L. (2017). *Probabilidade: aplicações à Estatística*. Livros Técnicos e Científicos Editora Ltda, Rio de Janeiro, 2nd edition.
- Ross, S. (2019). *A first course in probability*. Pearson Education, Malaysia, 10th edn. global edition.



Thank you!

✉ tiago.magalhaes@ufjf.br

📄 ufjf.br/tiago_magalhaes

🌐 Department of Statistics, Room 319

