

Moment generating functions for obtaining distributions

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Juiz de Fora, September 30, 2026



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- 1 Introduction
- 2 Moment generating function
- 3 Cumulative distribution function
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Introduction

When defining a random variable (RV) X , we emphasize that X is a function defined on the sample space Ω that assigns a real number to each $\omega \in \Omega$.



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To illustrate the general case, consider a two-dimensional random vector $(X, Y)^\top$, consisting of a pair of functions,

$$X = X(\omega), Y = Y(\omega),$$

each of which is defined on the sample space of some experiment and assigns a real number to every $\omega \in \Omega$,



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The question that arises is the following: given the joint probability distribution of $(X, Y)^{\top}$, what is the probability distribution of $Z = H(X, Y)$?



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There are situations in which the procedure for finding the probability distribution of a function of a random vector can be simplified.

One such situation occurs when the random variables that compose the random vector are independent and the **moment generating function** or the **cumulative distribution function** is known.



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Moment generating function

The moment generating function (MGF) is an important tool in the study of various aspects of probability distributions.

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Moment generating function

Thus,

$$M_Y(t) = \mathbb{E}[\exp(tY)] = \mathbb{E}\left[\exp\left(t \sum_{i=1}^n X_i\right)\right] = \mathbb{E}\left[\prod_{i=1}^n \exp(tX_i)\right].$$



Moment generating function

Now, suppose that $X_1, X_2, \dots, X_n$ are independent random variables,

$$M_Y(t) = \mathbb{E} \left[\prod_{i=1}^n \exp(tX_i) \right] \stackrel{\text{Ind.}}{=} \prod_{i=1}^n \mathbb{E} [\exp(tX_i)] = \prod_{i=1}^n M_{X_i}(t).$$

That is, the MGF of Y is the product of the MGFs of the RVs $X_1, X_2, \dots, X_n$.



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Moment generating function

If, in addition, the n random variables $X_1, X_2, \dots, X_n$ are identically distributed, we have:

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Examples

Example 1. If X and Y are two independent RVs with Poisson distributions with parameters λ_1 and λ_2 , respectively, determine the probability distribution of $Z = X + Y$.



Examples

Example 2. Suppose that $Z = 2W + X - Y$, where W , X and Y are independent random variables, each with $N(\mu_W, \sigma_W^2)$, $N(\mu_X, \sigma_X^2)$ and $N(\mu_Y, \sigma_Y^2)$, respectively.

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Cumulative distribution function

The cumulative distribution function, like the MGF, uniquely determines the probability distribution; therefore, it can also be used to determine the probability distribution of an RV that is a function of other RVs.



Cumulative distribution function

We will return to this topic in the lecture on **Order statistics**.



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Thank you!

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