

Multivariate distributions

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Outline

- 1 Multinomial distribution
- 2 Multivariate normal distribution
- 3 Wishart distribution
- 4 Dirichlet distribution
- 5 References



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The probability function of a multidimensional discrete random variable $\mathbf{X} \sim \text{multinomial}(n, \mathbf{p})$ is given by:

$$p(x_1, x_2, \dots, x_k) = \frac{n!}{x_1! x_2! \dots x_k!} p_1^{x_1} p_2^{x_2} \dots p_k^{x_k} \mathbb{I}_{\{n\}}(x_1 + x_2 + \dots + x_k),$$



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The expectation and variance of a multidimensional multinomial DRV $\mathbf{X}$ are given, respectively, by $\mathbb{E}(\mathbf{X}) = [\mathbb{E}(X_1), \mathbb{E}(X_2), \dots, \mathbb{E}(X_k)]^\top$ and

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Bivariate normal distribution

Let $(X_1, X_2)^\top$ be a two-dimensional continuous random variable (CRV), taking all values in the Euclidean plane. We say that $(X_1, X_2)^\top$ has a bivariate normal distribution if its joint probability density function (PDF) is given by the following expression



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where $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_k)^\top \in \mathbb{R}^k$ is the vector of means, $\boldsymbol{\Sigma} \in \mathbb{R}^{k \times k}$ is the covariance matrix (a positive definite matrix), of dimension $k \times k$



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$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

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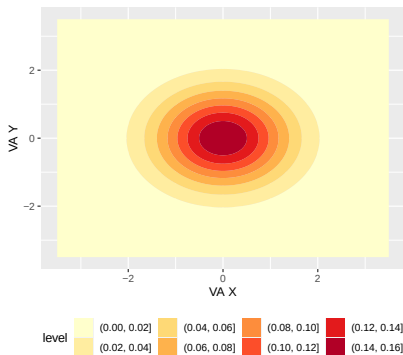
$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$



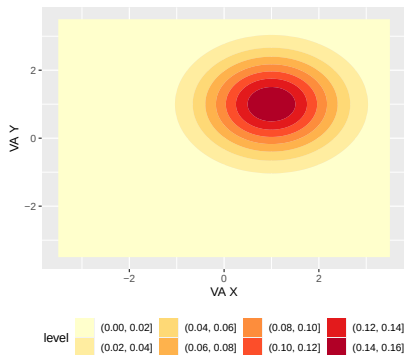
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Examples:

- ① $\mu_1 = 0, \mu_2 = 0, \sigma_1^2 = 1, \sigma_2^2 = 1$ and $\rho = 0$;
- ② $\mu_1 = 1, \mu_2 = 1, \sigma_1^2 = 1, \sigma_2^2 = 1$ and $\rho = 0$;
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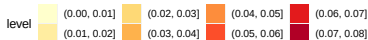
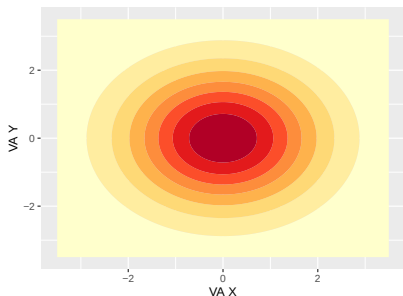


(a) Example 1.

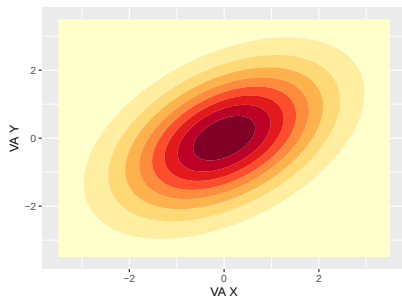


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Figure 1: Contour plot of bivariate normal distributions.



(a) Example 3.



(b) Example 4.

Figure 2: Contour plot of bivariate normal distributions.

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- 1 Multinomial distribution
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- 3 Wishart distribution**
- 4 Dirichlet distribution
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Wishart distribution

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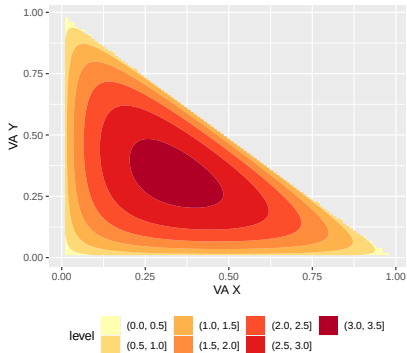
Examples:

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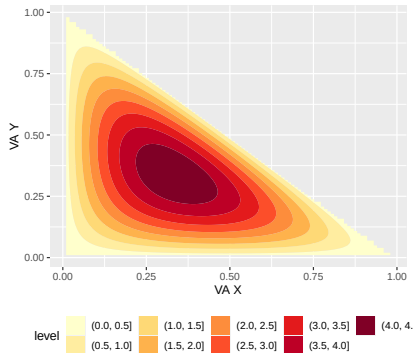
② $\lambda_1 = 2$, $\lambda_2 = 2$ and $\lambda_3 = 2$;

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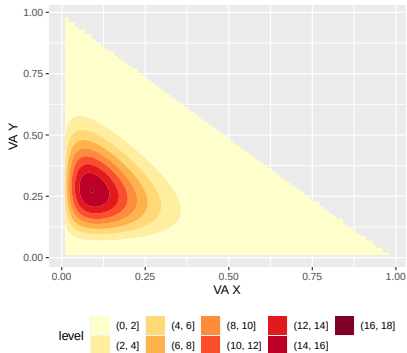


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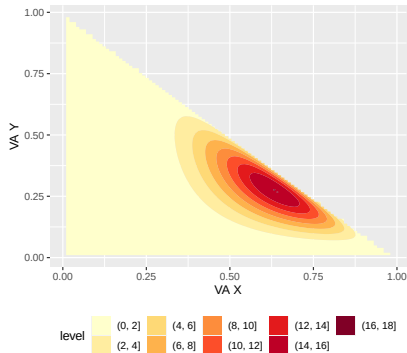


(b) Example 2.

Figure 3: Contour plot of three-dimensional Dirichlet distributions.



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(b) Example 4.

Figure 4: Contour plot of three-dimensional Dirichlet distributions.

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Thank you!

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