

Generating functions

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Juiz de Fora, September 23, 2026



Outline

- 1 Generating Functions
- 2 Multidimensional Moment Generating Function
- 3 Bibliography



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Probability Generating Function

The probability generating function (PGF) of the discrete random variable (RV) X , taking values in $\{0, 1, 2, \dots\}$, $P_X(t)$, is defined by

$$P_X(t) = \mathbb{E} [t^X] = \sum_{x=-\infty}^{+\infty} t^x p_X(x),$$

for $t \in \mathbb{R}$.



Moment Generating Function

The moment generating function (MGF) of the RV X , $M_X(t)$, is defined by

$$M_X(t) = \mathbb{E} \left[e^{tX} \right] = \begin{cases} \sum_{x=-\infty}^{+\infty} e^{tx} p_X(x), & \text{discrete case,} \\ \int_{-\infty}^{+\infty} e^{tx} f_X(x) dx, & \text{continuous case,} \end{cases}$$

for those $t \in \mathbb{R}$ for which the expectation is finite.

Characteristic Function

The PGF and MGF are very useful tools. However, they do not always exist. Therefore, it is interesting to obtain a function that always exists.



Characteristic Function

A complex RV X can be written as:

$$X = X^a + iX^b,$$

where $i = \sqrt{-1}$, and X^a and X^b are real RVs.



Characteristic Function

Let X be a real-valued RV. The characteristic function of X , $\varphi_X(t)$, is defined by

$$\varphi_X(t) = \mathbb{E} \left[e^{itX} \right] = \mathbb{E} [\cos(tX)] + i\mathbb{E} [\sin(tX)],$$

for $t \in \mathbb{R}$ and $i = \sqrt{-1}$.



Characteristic Function

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- ④ If the RVs X and Y are independent, then $\varphi_{X+Y}(t) = \varphi_X(t) \times \varphi_Y(t)$;



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- ⑥ $\varphi_X(t)$ also generates moments.



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Let X be an RV. When the MGF admits the corresponding complex extension, we have the following relationship:

- $M_X(t) = P_X(e^t)$, if X is a discrete RV;



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Cumulant Generating Function

The cumulant generating function (CGF) of the RV X , $K_X(t)$, is defined by

$$K_X(t) = \log M_X(t),$$

for those $t \in \mathbb{R}$ for which $M_X(t)$ is finite. When the corresponding complex extension is well-defined, the CGF can be defined as

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Multidimensional Moment Generating Function

The joint MGF of the two-dimensional RV $(X, Y)^\top$, $M(t_1, t_2)$, is defined for all real values t_1 and t_2 by

$$\begin{aligned} M(t_1, t_2) &= \mathbb{E} \left[e^{t_1 X + t_2 Y} \right] \\ &= \begin{cases} \sum_{x=-\infty}^{+\infty} \sum_{y=-\infty}^{+\infty} e^{t_1 x + t_2 y} p(x, y), & \text{discrete case,} \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{t_1 x + t_2 y} f(x, y) dx dy, & \text{continuous case.} \end{cases} \end{aligned} \quad (1)$$

If the joint MGF exists in a neighborhood of $(0, 0)$, it uniquely determines the joint distribution of $(X, Y)^\top$.



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Moment Generating Function

From the joint MGF in (1), we have

$$\left. \frac{\partial^{r+s} M(t_1, t_2)}{\partial t_1^r \partial t_2^s} \right|_{t_1=0, t_2=0} = \mathbb{E}(X^r Y^s),$$

whenever the corresponding moments exist.



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Moment Generating Function

For example, $r = s = 1$,

$$\left. \frac{\partial^2 M(t_1, t_2)}{\partial t_1 \partial t_2} \right|_{t_1=0, t_2=0} = \mathbb{E}(XY).$$

Moment Generating Function

If $r = 1$ and $s = 0$,

$$\left. \frac{\partial M(t_1, t_2)}{\partial t_1} \right|_{t_1=0, t_2=0} = \mathbb{E}(X).$$

If $r = 0$ and $s = 1$,

$$\left. \frac{\partial M(t_1, t_2)}{\partial t_2} \right|_{t_1=0, t_2=0} = \mathbb{E}(Y).$$



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If $r = 1$ and $s = 0$,

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If $r = 0$ and $s = 1$,

$$\left. \frac{\partial M(t_1, t_2)}{\partial t_2} \right|_{t_1=0, t_2=0} = \mathbb{E}(Y).$$



Moment Generating Function

From (1), we can obtain the marginal MGFs as follows

$$M_X(t) = \mathbb{E} \left[e^{tX} \right] = M(t, 0),$$

$$M_Y(t) = \mathbb{E} \left[e^{tY} \right] = M(0, t).$$



Moment Generating Function

If X and Y are independent RVs, then the MGF in (1) can be written as follows

$$M(t_1, t_2) = M_X(t_1) \times M_Y(t_2).$$



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Moment Generating Function

In general, the joint MGF of the multidimensional RV $(X_1, X_2, \dots, X_n)^\top$, $M(t_1, t_2, \dots, t_n)$, is defined by

$$M(t_1, t_2, \dots, t_n) = \mathbb{E} \left[e^{t_1 X_1 + t_2 X_2 + \dots + t_n X_n} \right].$$



Moment Generating Function

Example. Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ be independent RVs.

- ① What are the MGFs of X and Y , respectively? (No proof is required).
- ② What is the joint MGF of the two-dimensional RV $(X, Y)^\top$?
- ③ From the joint MGF, find $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\mathbb{E}(XY)$;
- ④ Calculate $\text{Cov}(X, Y)$.



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Thank you!

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