Regression analysis

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Outline

1. Motivation

2. Simple linear regression model

3. Application

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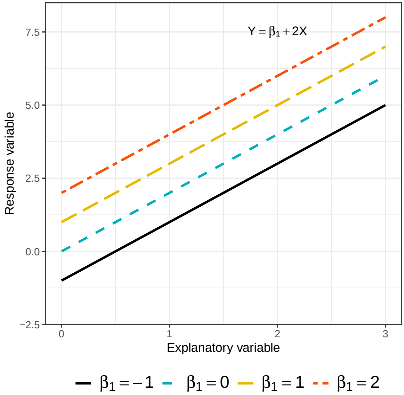
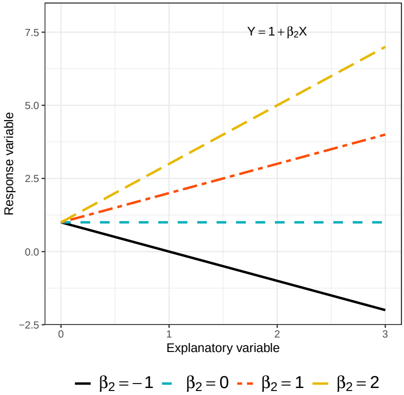


(a) Free β_1 and $\beta_2 = 2$.



(b) $\beta_1 = 1$ and free β_2.

Figure 1: Regression lines as a function of β_1 and β_2.

Linear regression model

In order to estimate β_1, β_2, and σ^2, and to make the relationship between the variables explicit, it is necessary to draw an independent sample of size n from the pair (Y, x), i.e., $(Y_1, x_1), (Y_2, x_2), \ldots, (Y_n, x_n)$,

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Linear regression model

Alternatively,

$$Y_\ell = \beta_1 + \beta_2 x_\ell + \varepsilon_\ell,$$

with $\varepsilon_\ell \sim \mathcal{D}(0, \sigma^2)$, $\ell = 1, 2, \ldots, n$. In other words, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are independent random variables with mean zero and common variance σ^2.

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Estimation

By the Least Squares Method, the estimators of β_1 and β_2 are given, respectively, by

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2\bar{X} \text{ e } \hat{\beta}_2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}. \quad (1)$$

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Variance estimation

The estimator of the variance, σ^2, is given by

$$\hat{\sigma}^2 = \frac{\sum_{\ell=1}^{n}\left(Y_\ell - \hat{\beta}_1 - \hat{\beta}_2 X_\ell\right)^2}{n-2}. \tag{2}$$

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Since 2003, each soccer team in the Brazilian men's football championship, the Brasileirão, plays the others twice (a double round-robin system) and, at the end, the club with the most points is crowned champion (Magalhães et al., 2025).

The Brasileirão was nominated as the World's Strongest National League in 2021 and 2022 by the International Federation of Football History & Statistics (IFFHS, 2022, 2023).

Application

Here, our interest may be in assessing whether there is a linear relationship between points percentage and goal difference among the champion teams.

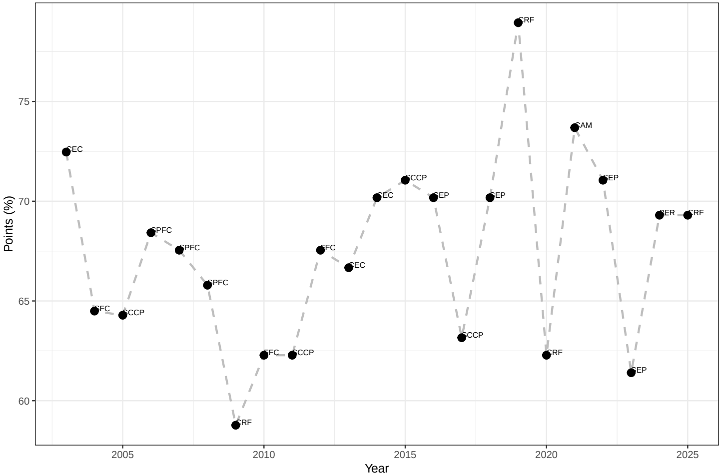

Figure 2: Points (%) of the champion teams over the years.

Application

Table 1: States of the champion teams.

Minas Gerais	Rio de Janeiro	São Paulo
4	7	12
(17.4%)	(30.4%)	(52.2%)
2	3	4

Application

Table 2: Points (%) of the champion teams.

Min	Q1	Md	Mean	Q3	Max	SD	CV
58.8	63.7	67.5	67.4	70.2	78.9	4.7	7.0

Application

Table 3: Points (%) of the champion teams, by state.

	Min	Q1	Md	Mean	Q3	Max	SD	CV
Minas Gerais	66.7	69.3	71.3	70.7	72.8	73.7	3.1	4.4
Rio de Janeiro	58.8	62.3	67.5	66.9	69.3	78.9	6.7	9.9
São Paulo	61.4	64.0	66.7	66.7	70.2	71.1	3.5	5.3

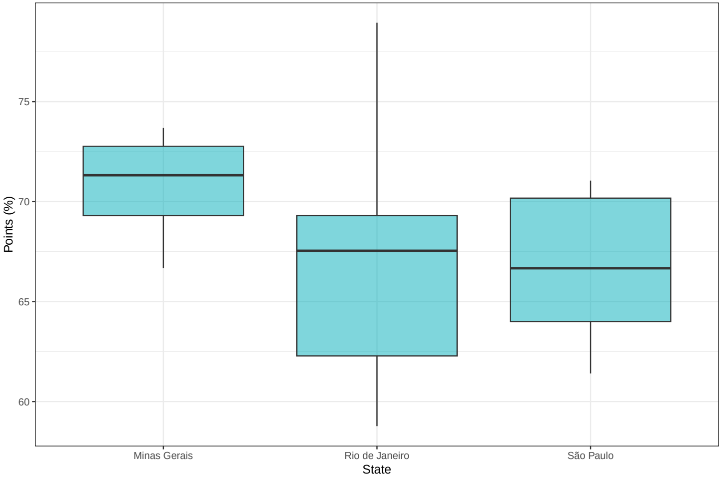

Figure 3: Boxplot of points (%) of the champion teams by state.

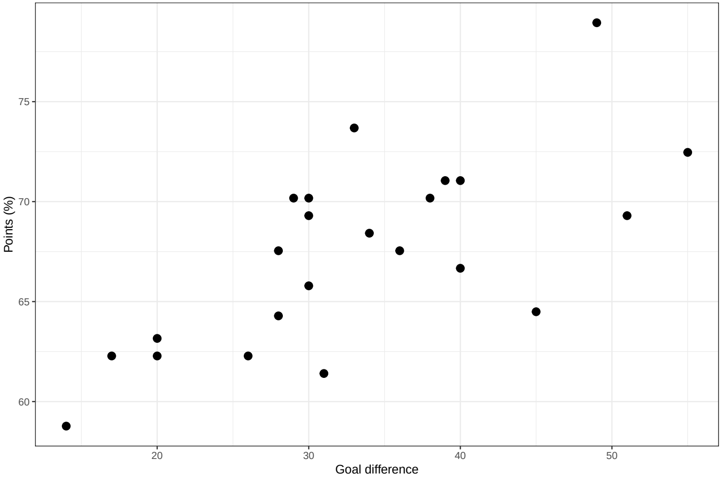

Figure 4: Scatterplot of points (%) and goal difference of the champion teams.

After fitting the model (parameter estimation), using (1) and (2), we have the following estimated model,

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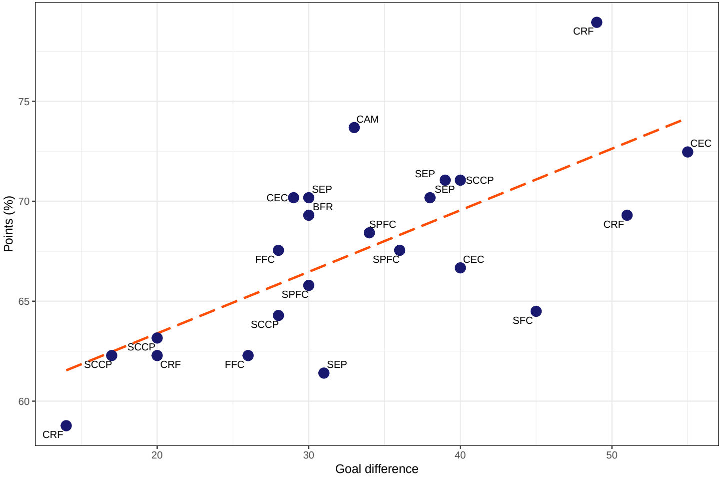

Figure 5: Scatterplot of points (%) and goal difference of the champion teams, with the estimated regression line.

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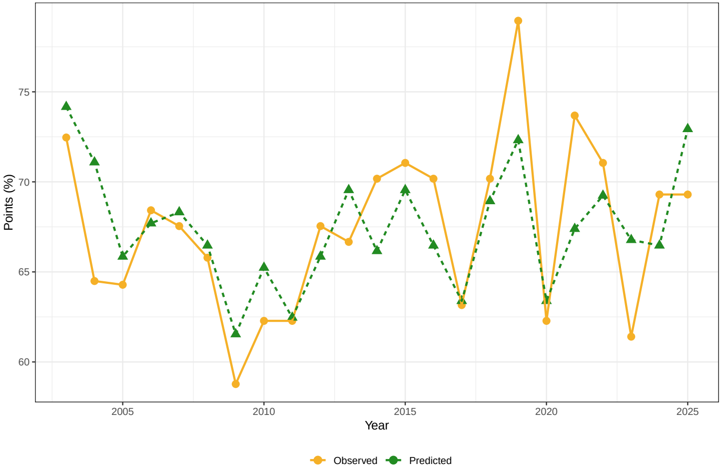

Figure 6: Observed and predicted points (%) from the model.

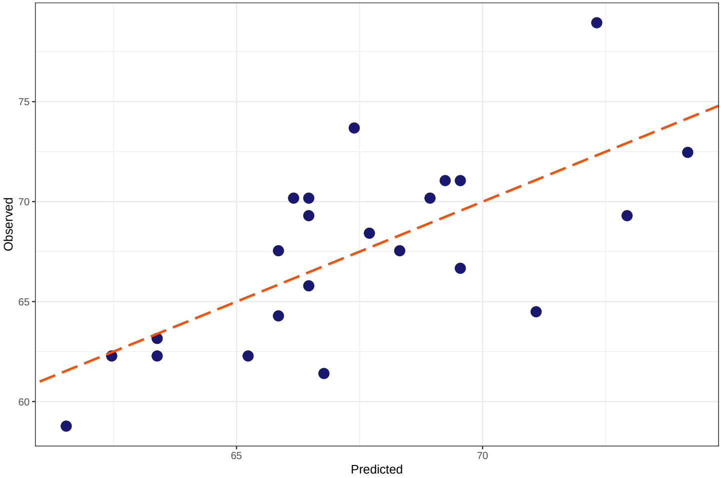

Figure 7: Observed and predicted points (%) from the model.

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Thank you!

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