Marginal and conditional distributions

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Outline

1. Marginal probability distributions
2. Conditional probability distributions
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For each bivariate random variable $(X, Y)^\top$, we associate two univariate random variables, namely X and Y, individually.

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The function p_X, defined for all possible values of X, is called the **marginal probability distribution** of X. Analogously, the function p_Y, defined for all possible values of Y, is called the **marginal probability distribution** of Y, where

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The PDFs in (1) correspond to the basic PDFs of the univariate random variables X and Y, respectively. For example,

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\mathbb{P}(c \leq X \leq d) &= \mathbb{P}(c \leq X \leq d,\ -\infty \leq Y \leq +\infty) \\
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Similarly, the conditional CDF of X given $Y = y$ is defined by

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Note that,

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Observation

Marginal or conditional PMFs and PDFs do not always have a closed-form expression.

Marginal and conditional probability distributions

Example 4. Let $Y|X = x \sim \text{Rayleigh}(x)$ and $X \sim \text{log-normal}(\mu, \sigma^2)$, where $\mu > 0$ and $\sigma^2 > 0$. That is,

$$f_{Y|X}(y|x) = \frac{y}{x^2} \exp\left\{-\frac{1}{2}\left(\frac{y}{x}\right)^2\right\} \mathbb{I}_{(0,\infty)}(y),$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\} \mathbb{I}_{(0,\infty)}(x).$$

Obtain the marginal PDF of Y.

Marginal and conditional probability distributions

The joint PDF of $(X, Y)^\top$ is given by

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\begin{aligned}
f(x, y) &= f_{Y|X}(y|x) \times f_X(x) \\
&= \frac{y}{\sqrt{2\pi}\sigma x^3} \exp\left\{-\frac{1}{2}\left[\left(\frac{\log x - \mu}{\sigma}\right)^2 + \left(\frac{y}{x}\right)^2\right]\right\} \mathbb{I}_{(0,\infty)}(x)\, \mathbb{I}_{(0,\infty)}(y).
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The marginal PDF of Y is obtained as follows:

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f_Y(y) &= \int_{-\infty}^{+\infty} f(x,y)\,dx \\
&= \int_{-\infty}^{+\infty} \frac{y}{\sqrt{2\pi}\sigma x^3} \exp\left\{-\frac{1}{2}\left[\left(\frac{\log x-\mu}{\sigma}\right)^2 + \left(\frac{y}{x}\right)^2\right]\right\} \mathbb{I}_{(0,\infty)}(x)\,\mathbb{I}_{(0,\infty)}(y)\,dx \\
&= \frac{y}{\sqrt{2\pi}\sigma} \int_{0}^{+\infty} \frac{1}{x^3} \exp\left\{-\frac{1}{2}\left[\left(\frac{\log x-\mu}{\sigma}\right)^2 + \left(\frac{y}{x}\right)^2\right]\right\} dx\,\mathbb{I}_{(0,\infty)}(y). \qquad (3)
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The integral in (3) does not have a closed-form expression. Therefore, (3) gives the marginal PDF of Y.

Result

When $Y|X = x \sim \text{Rayleigh}(x)$ and $X \sim \text{log-normal}(\mu, \sigma^2)$, the marginal distribution of Y is known as the Suzuki distribution (Suzuki, 1977), with parameters μ and σ^2.

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Thank you!

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