Random vectors

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Juiz de Fora, August 21, 2026

Outline

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let n functions,

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Thus, we will refer to $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^\top$ as an n-dimensional, multidimensional, multivariate random variable, or, more precisely, a **random vector**.

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Examples

Example 1. Suppose that the two-dimensional continuous random variable $(X, Y)^\top$ has joint PDF given by

$$f(x, y) = \left(x^2 + \frac{xy}{3}\right) \mathbb{I}_{[0,1]}(x) \, \mathbb{I}_{[0,2]}(y). \tag{1}$$

1. Show that $f(x, y)$ in (1) is a joint PDF.
2. Calculate $\mathbb{P}(X > 0.5, Y > 1)$.
3. Calculate $\mathbb{P}(B)$, where $B = \{X + Y \geq 1\}$.

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Bibliography

Magalhães, M. N. (2015), *Probabilidade e variáveis aleatórias*, 3 edn, Editora da Universidade de São Paulo, São Paulo.

Meyer, P. L. (2017), *Probabilidade: aplicações à Estatística*, 2 edn, Livros Técnicos e Científicos Editora Ltda, Rio de Janeiro.

Ross, S. (2019), *A first course in probability*, 10 edn. global edn, Pearson Education, Malaysia.

Thank you!

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