

Fundamental Concepts

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Juiz de Fora, August 19, 2026



Outline

- 1 Motivation
- 2 Experiment
- 3 Probability
- 4 References



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Kolmogorov, A. N. (1933). *Grundbegriffe der Wahrscheinlichkeitsrechnung*. Springer, Berlin.



Thank you!

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