Definitions and Assumptions

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Juiz de Fora, August 28, 2026

Outline

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$$\mu_\ell = \mathbb{E}(Y_\ell) = \beta_1 x_{\ell 1} + \beta_2 x_{\ell 2} + \cdots + \beta_p x_{\ell p},$$

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where $\boldsymbol{x}_\ell = (x_{\ell 1}, x_{\ell 2}, \ldots, x_{\ell p})^\top$ are known variables (quantities), $\beta = (\beta_1, \beta_2, \ldots$

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We can assume that

$$Y_\ell = \beta_1 x_{\ell 1} + \beta_2 x_{\ell 2} + \cdots + \beta_p x_{\ell p} + \varepsilon_\ell, \tag{1}$$
$$= \boldsymbol{x}_\ell^\top \boldsymbol{\beta} + \varepsilon_\ell,$$

where $\varepsilon_\ell \sim \mathcal{D}(0, \sigma^2)$, $\ell = 1, 2, \ldots, n$, and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are independent random variables with mean zero and common variance σ^2.

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Interpretation of the model parameters

In the LRM, the vectors $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)^\top$ and $\boldsymbol{x}_\ell$, $\ell = 1, 2, \ldots, n$, are called the response vector and predictor vector, respectively, and the parameters belonging to the vector β are called regression coefficients.

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The matrix $\boldsymbol{X}$ has n rows (sample size) and p columns (number of predictor variables, covariates), of rank p ($n \geq p$), and $\boldsymbol{X}$ is called the design matrix.

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Normality

For classical finite-sample inferential procedures to be applied, we need to assume that $\varepsilon_\ell \sim \mathcal{N}(0, \sigma^2)$, $\ell = 1, 2, \ldots, n$ and $\boldsymbol{\varepsilon} \sim \mathcal{N}_n(\mathbf{0}, \sigma^2 \boldsymbol{I}_n)$, in (1) and (3), respectively. With this assumption, the LRM is called the **normal linear regression model** (NLM).

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Note that, $\varepsilon_\ell \sim \mathcal{N}(0, \sigma^2) \Rightarrow Y_\ell \sim \mathcal{N}(\boldsymbol{x}_\ell^\top \beta, \sigma^2)$, $\ell = 1, 2, \ldots, n$ and $\boldsymbol{\varepsilon} \sim \mathcal{N}_n(\mathbf{0}, \sigma^2 \boldsymbol{I}_n) \Rightarrow \boldsymbol{Y} \sim \mathcal{N}_n(\boldsymbol{X}\beta, \sigma^2 \boldsymbol{I}_n)$.

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Example 1. Westwood Company example (Neter et al., 1983, p. 36), also available in Table 1.

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Table 1: Data on lot size and number of man-hours.

Production run	1	2	3	4	5	6	7	8	9	10
Lot size	30	20	60	80	40	50	60	30	70	60
Man-hours	73	50	128	170	87	108	135	69	148	132

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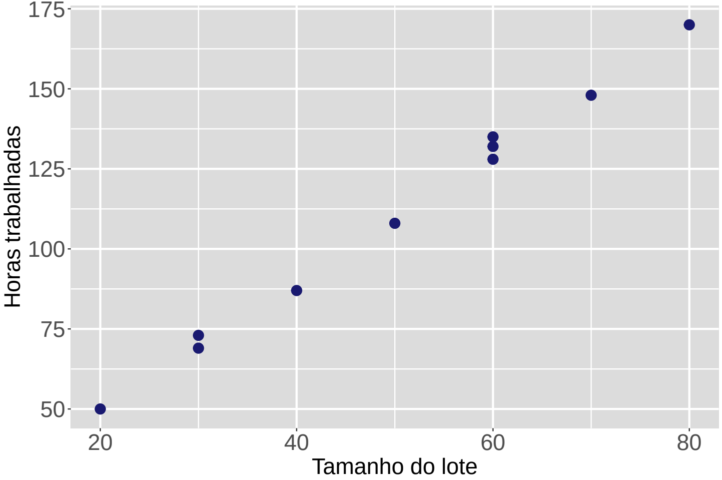

Figure 1: Scatter plot for the Westwood Company example.

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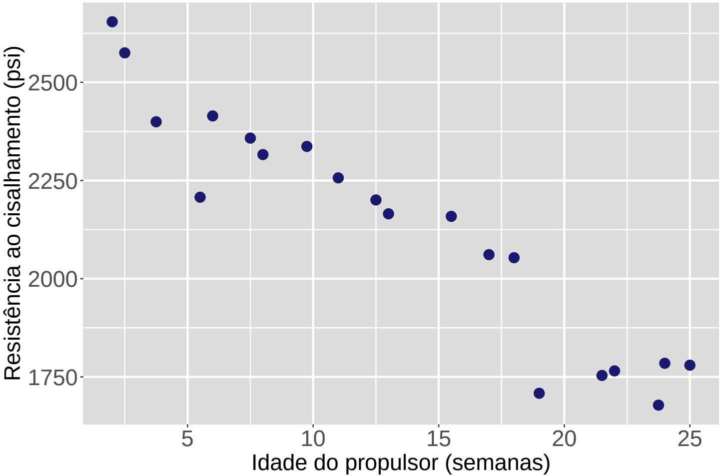
Shear strength (psi)	Age of propellant (weeks)	Shear strength (psi)	Age of propellant (weeks)
2158.70	15.50	2165.20	13.00
1678.15	23.75	2399.55	3.75
2316.00	8.00	1779.80	25.00
2061.30	17.00	2336.75	9.75
2207.50	5.50	1765.30	22.00
1708.30	19.00	2053.50	18.00
1784.70	24.00	2414.40	6.00
2575.00	2.50	2200.50	12.50
2357.90	7.50	2654.20	2.00
2256.70	11.00	1753.70	21.50



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Unit normal scaling

In the second possibility, called **unit normal scaling**, the procedure is as follows:

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$$\boldsymbol{X}_{\mathrm{Sit.1}} = \begin{bmatrix} 1 & x_{12} & 0 \\ \vdots & \vdots & \vdots \\ 1 & x_{k2} & 0 \\ 1 & x_{(k+1)2} & 1 \\ \vdots & \vdots & \vdots \\ 1 & x_{n2} & 1 \end{bmatrix} \text{ and } \boldsymbol{X}_{\mathrm{Sit.2}} = \begin{bmatrix} 1 & x_{12} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{k_1 2} & 0 & 0 \\ 1 & x_{(k_1+1)2} & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{k_2 2} & 0 & 1 \\ 1 & x_{(k_2+1)2} & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n2} & 1 & 0 \end{bmatrix}.$$

Qualitative covariates

If a predictor variable has k categories, $k - 1$ dummy variables will be required.

Examples

Example 3. Dataset from Walker (1962). The author studied the sounds of male crickets in the USA, where he measured the number of pulses per second of 31 crickets at different temperatures, also available in Table 3.

Table 3: Cricket pulse rate.

Spec.	Temp.	Pulses	Spec.	Temp.	Pulses	Spec.	Temp.	Pulses
ex	20.8	67.9	ex	29.0	100.8	niv	21.0	58.9
ex	20.8	65.1	ex	30.4	99.3	niv	22.1	60.7
ex	24.0	77.3	ex	30.4	101.7	niv	23.5	69.8
ex	24.0	78.7	niv	17.2	44.3	niv	24.2	70.9
ex	24.0	79.4	niv	18.3	47.2	niv	25.9	76.2
ex	24.0	80.4	niv	18.3	47.6	niv	26.5	76.1
ex	26.2	85.8	niv	18.3	49.6	niv	26.5	77.0
ex	26.2	86.6	niv	18.9	50.3	niv	26.5	77.7
ex	26.2	87.5	niv	18.9	51.8	niv	28.6	84.7
ex	26.2	89.1	niv	20.4	60.0			
ex	28.4	98.6	niv	21.0	58.5			

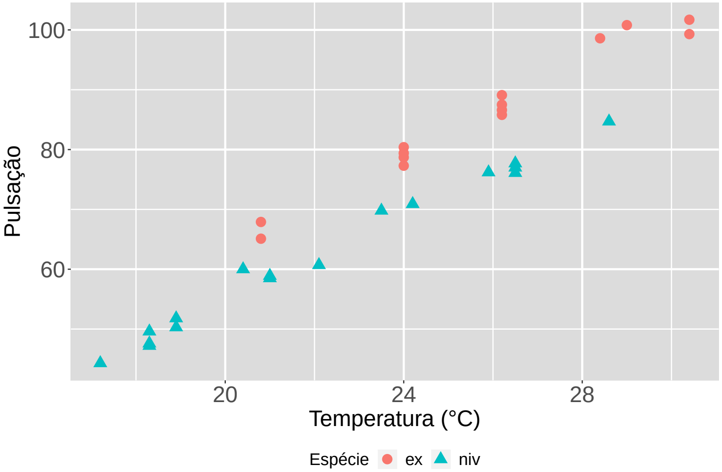

Figure 3: Scatter plot of the number of pulses per second versus temperature.

Examples

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5. With the 5,000 replications, we calculate the mean of the 5,000 covariances.

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Simulation study

Table 4: Results.

n	Covariance	SD
10	0.063	0.167
20	0.090	0.137
40	0.068	0.085
80	0.086	0.066
160	0.088	0.047

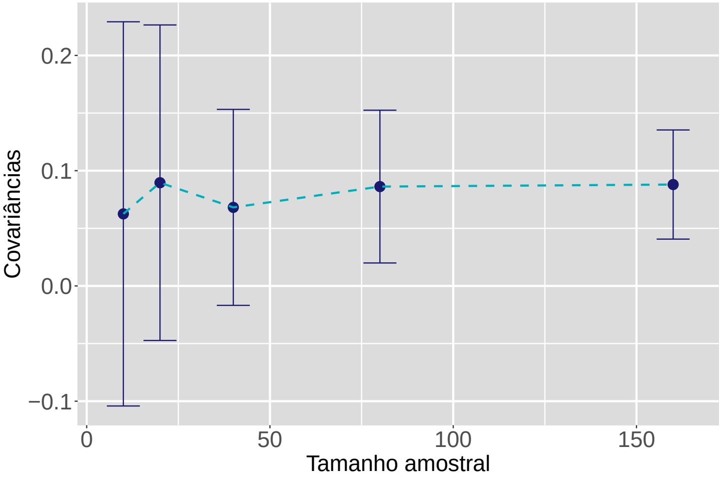

Figure 4: Covariances with respect to sample size.

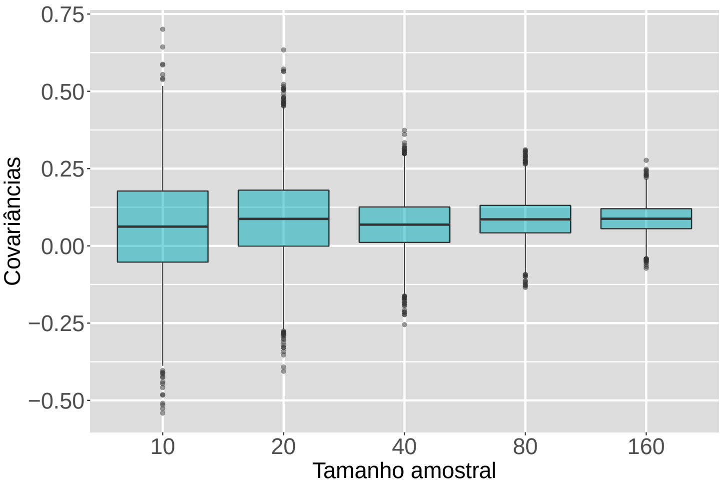

Figure 5: Boxplot of covariances by sample size.

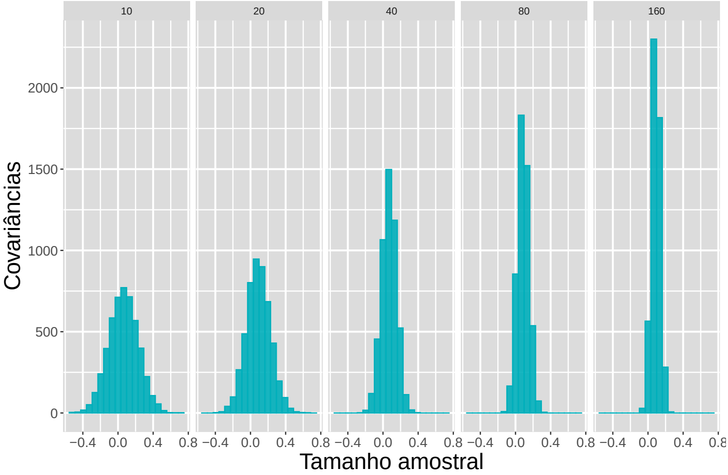

Figure 6: Histogram of covariances by sample size.

Simulation study

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Outline

Final remarks

For a better understanding of the concepts of linear regression models, see Chapter 1 of Kutner et al. (2005) and Montgomery et al. (2021).

Outline

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Thank you!

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