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CAPTURING ASYMMETRY IN REAL EXCHANGE RATE WITH QUANTILE AUTOREGRESSION

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Capturing asymmetry in real exchange rate with quantile autoregression[1]

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Abstract

Quantile autoregression is used to explore asymmetries in the adjustment process of pair wise real exchange rate between the Italian lire, French franc, Deutsch mark, and the British pound. Based on the best specification for each quantile we construct predicted conditional density functions which guided us to identify two sources of asymmetry: 1) dispersion depends on the conditioned value of the real exchange rate, i.e., "*conditional*" heterokedasticity; 2) the probability of increases and falls also changes according to the conditioned value, i.e., there is higher probability for the real exchange rate to appreciate (depreciate) given the currency is depreciated (appreciated).

We only verified strong heterokedasticity in relations among the lire, franc, and mark, which was resolved by estimating quadratic autoregressive model for some quantiles. Relations involving the pound presented stable but higher dispersion indicating larger probability of wider oscillation.

Keywords: Exchange Rate, Quantile Regression, Purchasing Power Parity, Asymmetry

JEL Classification: C14, C22, F31

[1] This is a forthcoming article in the Applied Economics.

I. Introduction

The validity of the purchasing power parity (PPP) is an essential part of most international macroeconomics and finance models. A necessary condition for validating the PPP is that the real exchange rate (RER) should feature a mean reverting process. Despite the consensus on its inability to explain short run movements in the real exchange rate, mostly driven by shocks on the nominal exchange rate, economic rationale suggests that in longer horizon the RER should converge to its equilibrium value preceding the shock.

Works using secular data on real exchange rate[2] support the PPP as a long run theory, but Rogoff (1996) still observed the difficulty to reconcile the immense short term volatiliy of the RER with the very low rate at which shocks seem to damp out. This fact was called, by Rogoff, as the PPP puzzle. A possible nonlinear adjustment of the RER has been considered the most important fact to explain the puzzle, encountering support in theoretical models that incorporate transaction costs for trading[3]. As a result of such models, a RER can follow a random walk in period *t* if it assumes central values in $t-1$, but behaves as a convergent AR process if it takes extreme values in $t-1$.

Trying to identify if this type of asymmetric adjustment better explains the pattern followed by real exchange rates became the next logical step in this literature, since it would validate the transaction cost theory explanation for the PPP puzzle. For

[2] Frankel (1986), Edison (1987), Glen (1992), Lothian and Taylor (1996).

[3] Other explanations can also be reconciled with nonlinearity asymmetry in real exchange rates. In a very complete survey on PPP, Taylor (2003) mentions another two theoretical explanations that could justify a nonlinear adjustment in real exchange rates. One has to do with the interaction of heterogeneous agents in the foreign exchange market. Another hypothesis has to do with the effects of official intervention in the foreign exchange market.

this matter, and relying mostly on threshold autoregression (TAR) models, Michael, Nobay, and Peel (1997), Obstfeld and Taylor (1997), Bec, Carrasco and Salem (2004), Leon and Najarian (2005), among others, found evidences supporting non linearity in several RER relations. From the statistical point of view, these findings also explained why traditional unit root tests tended not to reject the random walk in real exchange rate series. The main problem was in the misspecification of the alternative hypothesis, which normally imposed a linear AR process for the RER. Due to this misspecification, results indicated that that PPP is a flawed theory even in the long run.

In this paper we innovate by exploring asymmetry in real exchange rates using quantile autoregression (QAR), as developed by Koenker and Xiao (2002, 2004). This allows for a deeper understanding of the process followed by each RER, since quantile regression, estimated at several percentiles, permits characterizing the entire conditional distribution of each time series. In this sense our approach sheds a more complete light on this debate. Our conclusions are reached based on the behavior of pair wise real exchange rate between the Italian lira, French franc, Deutsch mark, and the British pound for a sample ranging from January of 1973 to December of 1998.

The main contribution of this paper is the identification and measurement of two sources of asymmetry in the adjustment process of RER. The first is the heterokedasticity: dispersion, measured in terms of standard deviation and range, varies with the conditioned value of the real exchange rate. The second has to do with the probability of the real exchange rate to increase and fall, which changes according to the conditioned value, especially when conditioned at extreme values. We found, for instance, a higher probability for the real exchange rate to appreciate (depreciate) given the currency is depreciated (appreciated), which favors the theoretical models resulting in nonlinear adjustment of the real exchange rate.

These findings were only possible because QAR, differing from other methods, permits the characterization of the entire distribution of a time series, allowing for a complete description of its stochastic process with great flexibility. To illustrate this point, we were able to choose different specification for each estimated quantile function; a linear model was the best fit for some quantiles, while a quadratic specification better fitted others. Based on the best specification for each quantile we estimated conditional density functions and the probabilities for movements in the RER. Other procedures do not present such flexibility, not allowing a very detailed characterization of the data generating process.

The finding that the RER behaves in an asymmetric way, however, does not imply in symmetric thresholds. For this matter, we observe that being equidistant from a central conditioned value, but in opposite directions, not necessarily result in similar probabilities form moving up or down, supporting the use of non symmetric TAR models, as implemented by Leon and Najarian (2005).

QAR also makes it possible carrying unit root test at each quantile which helps in the assessment of local and global persistent process, allowing, in this way, comparison to several other procedures that try to capture these patterns in a time series.

Looking specifically at each currency, the heterokedasticity was only identified for relations between the lira, franc, and mark. Dispersion for RER involving the pound are larger but stable, meaning that estimated standard deviation is invariant to the conditioned value used to predict the real exchange rate.

The rest of this paper proceeds as follows. In the next section we review recent empirical literature on the behavior of the real exchange rate. In section 3 we briefly discuss quantile autoregression developed by Koenker and Xiao (2002, 2004). Section 4 brings the results and analysis, while 5 concludes. In the appendix we plotted some predicted conditional density functions to help visualizing the asymmetric behavior followed by each RER time series.

II. Evidences on the Behavior of the Real Exchange Rate

The purchasing power parity (PPP) theory is based on the validity of the following equation: $P_t = E_t P_t^*$, where P_t and P_t^* refer to domestic and foreign price levels, respectively, and E_t is the nominal exchange rate between the two currencies (or the home price of the foreign currency). This relation says that a devaluation in the home currency (increase of E_t) will be reflected by similar increase of the domestic price level, P_t, and/or by a reduction in the foreigner's, P_t^*. If this is indeed the case, one should expect a constant real exchange rate; $q_t = E_t P_t^* / P_t$.

It is well known, however, that due to price stickiness, at least in the short run the real exchange rate is influenced almost entirely by variations in the nominal exchange rate, which implies in oscillation of q_t over time and failure of the PPP in the short run. Over a longer horizon deviations from equilibrium should disappear as prices start to adjust. This is the same as saying that a real exchange rate series should not feature a unit root. Based on this economic rationale, several works tried to verify if q_t behaves like a stationary time series.

The first tests for the validity of the PPP were based on the Augmented Dickey Fuller (ADF) type of equation which features the following specification:

$$q_t = \alpha_0 + \alpha_1 q_{t-1} + \sum_{j=1}^{p} \alpha_{j+1} \Delta q_{t-j} + u_t \qquad (1)$$

where $u_t \sim iid(0, \sigma^2)$ is the disturbance term. The validity of the PPP in the long run implies $|\alpha_1| < 1$.

Rejection of the null hypothesis of a unit root was hardly obtained, implying a random walk behavior for *RER* and therefore failure of the PPP theory[4]. It is known, however, that the low power of ADF and Phillips-Perron (PP) unit root tests makes it hard to distinguish between $\alpha_1 = 1$ from α_1 just close to 1. And indeed the estimated values for α_1 were all very close to 1[5].

The use of more powerful unit root tests using panel regression models delivered ambiguous results, sometimes favoring the stationarity of the *RER*[6]and some others favoring a random walk[7] behavior.

More recently, the literature has considered another two types of tests. One is related to the close to unity behavior of the *RER* and has been addressed by Kim and Lima (2004). They argued that the *RER* may be better described by a local persistent process, which postulates a great similarity with the unit root in the short run, but that would present a convergent behavior in longer horizons. They applied the Lima-Xiao (2002) test to capture this local persistent process and did not reject the hypothesis that the *RER* of the G7 countries has a root near to unit, but not exactly 1.

The other direction pursued by this literature is the incorporation of a non-linear adjustment process for the RER, which has been analyzed with threshold autoregressive (TAR) models. The economic intuition for the use of this model is that

[4] The constant non rejection of the unit root has been considered one of the six major puzzles in the international finance literature (Obstfeld and Rogoff, 2000).
[5] . See, for example, Darby (1980), Enders (1988) and Mark (1990).
[6] Froot and Rogoff (1995); Frankel and Rose (1996); Wu and Wu (2001); Papell (2002).
[7] O 'Connell (1996); Engel (1996); Canzoneri, Cumby, and Diba (1996).

transaction costs may create a region (called *band of inaction*) where market arbitrage is non-profitable, justifying the random walk behavior for the RER. However, if the RER is smaller than a lower threshold or greater than an upper one, international trade would be profitable causing the RER to behave like a stationary autoregressive process.

The work of Michael, Nobay and Peel (1997), Taylor, Peel and Sarno (2001), Bec, Carrasco and Salem (2004), and Leon and Najarian (2005) indicated that a three regime TAR better describes the stochastic process followed by several RER, corroborating the transaction cost theory. An important difference between the work of Leon and Najarian resides on the fact that they do not impose symmetric thresholds, which they found to be an important restriction.

Among the previous TAR work, only Bec, Carrasco and Salem (2004) used the same data we do in the current article. Based on their findings, the pair wise RER between the French Franc, Italian Lira, and the Deutsch Mark are better characterized by a three regime TAR process, since they rejected the null of unit root in favor of the TAR process. Quantile autoregression has not yet been considered to analyze the dynamics of a RER process. By filling this gap we obtain results not explored under previous econometric techniques.

III. Quantile Autoregression

In this section we briefly present the quantile autoregressive process as developed by Koenker and Xiao (2002, 2004). Let the autoregressive process of any time series y_t be represented by

$$y_t = \alpha_1 y_{t-1} + u_t, \; t = 1, \ldots, n. \tag{2}$$

and denote the τth quantile of u as $Q_u(\tau)$. Let $Q_{y_t}(\tau \mid \mathrm{y}_{\text{t-1}})$ be the τth conditional quantile of y_t conditional on y_{t-1} which can be represented as

$$Q_{y_t}(\tau \mid \mathrm{y}_{\text{t-1}}) = Q_u(\tau) + \alpha y_{t-1}. \tag{3}$$

Let $\alpha_0(\tau) = Q_u(\tau)$, $\alpha_1(\tau) = \alpha$ and define $\alpha(\tau) = (\alpha_0(\tau), \alpha_1(\tau))^T$ and $x_t = (1, y_{t-1})^T$. The previous equation can be rewritten as

$$Q_{y_t}(\tau \mid \mathrm{y}_{\text{t-1}}) = x_t^T \alpha(\tau). \tag{4}$$

The quantile autoregressive parameter $\alpha(\tau)$ is estimated according to the linear programming problem suggested by Koenker and Basset (1978). Each solution $\hat{\alpha}(\tau)$ is the τth autoregressive quantile coefficient. Given $\hat{\alpha}(\tau)$, the τth quantile function of y_t, conditional on the past information, can be estimated by

$$\hat{Q}_{y_t}(\tau \mid x_{t\text{-}1}) = x_t^T \hat{\alpha}(\tau), \tag{5}$$

while the conditional density of y_t can be estimated by the following difference quotients

$$\hat{f}_{y_t}(\tau \mid x_t) = \frac{(\tau_i - \tau_{i-1})}{\hat{Q}_{y_t}(\tau_i \mid x_t) - \hat{Q}_{y_t}(\tau_{i-1} \mid x_t)} \tag{6}$$

Now, suppose we need a more extensive model, similar, for instance, to the ADF type of equation:

$$y_t = \alpha_1 y_{t-1} + \sum_{j=1}^{p} \alpha_{j+1} \Delta y_{t-j} + u_t\ . \tag{7}$$

We can still have a quantile autoregression representation of this previous ADF equation. This is obtained by letting $\alpha_j(\tau) = \alpha_j$, $j = 1, \ldots, p+1$, after which we can define $\alpha(\tau) = (\alpha_0(\tau), \alpha_1(\tau), \ldots, \alpha_{p+1}(\tau))$ and $x_t = (y_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-q})^T$, leading to

$$Q_{y_t}(\tau \mid \Im_{t\text{-}1}) = x_t^T \alpha(\tau), \tag{8}$$

where $\Im_t$ is the σ-field generated by $\{y_s, s \le t\}$, and $Q_{y_t}(\tau \mid \Im_{t-1})$ is the τth conditional quantile of y_t, conditional on $\Im_{t-1}$. This representation is referred to as QAR(p) model.

Equation (8) can be used to test for several local behavior of the stochastic process followed by y_t, which may possess very different local behavior: for some quantiles it may pose a convergent pattern, while for others it may behave as a random walk or even as an explosive random variable.

The flexibility of the QAR(p) allows not only capturing this local properties. We can also use the model to gain long run perspectives on a time series. For instance, even if y_t behaves explosively at some quantiles, it can still be stationary in the long run, as far as $E(\alpha)^2 < 1$[8]. This condition can be verified by firstly computing $\int_{\varepsilon}^{1-\varepsilon} \alpha(\tau)^2 \, d\tau$ and then checking whether it is smaller than 1[9].

In the next section we will be using equations 7 and 8 to estimate autoregressive equations at several percentiles, a crucial step to characterize the entire condition distribution of several real exchange rate relations. We also report values of $\int_{\varepsilon}^{1-\varepsilon} \alpha(\tau)^2 \, d\tau$, in an attempt to gain insights on the long run behavior of these time series.

INSERT FIGURE1

IV. Estimation and Results

[8] Further details about this condition can be found in the corollary 2.1 of Koenker and Xiao (2002).

[9] A formal test for this condition has not yet been developed, but by computing this integral we can at least have an idea of how close the statistic is to unity.

The data was obtained from the IMF International Financial Statistics. The nominal exchange rate is the end of period, and the price deflator is the CPI. The logarithm of each RER series can be visualized in Figure 1.

Following the literature, we set a sample to capture the behavior of real exchange rates during the floating period, just after the end of Bretton Wood – January 1973 until the last month prior to the introduction of the Euro (December 1998).

If nonlinearity in RER is indeed mainly caused by transaction costs, choosing currencies of neighbor countries and integrated economies may minimize the impact of such costs in driving large differences in our results. For instance, since Germany, France and Italy decided to give up their local currencies to adopt the Euro, we expected to observe more similarities in the statistical behavior among the pairs of exchange rates involving these currencies than if compared to the exchange rate pairs involving the British pound. Discrepancies in the behavior of real exchange rates due to very distinguished business cycles may also be minimized, since the four countries are very integrated, leading them to be hit in a higher probability by similar shocks than what one should expect if very distant economies were compared.

INSERT TABLE 1

Unit Root Tests[10]

Two standards unit root tests (Augmented Dickey-Fuller and Phillips-Perron) were initially applied to our series. The lag order of the ADF equation, chosen according to the BIC criteria, was used in the OLS and also in the quantile regressions.

INSERT TABLE 2

[10] For a detailed description of quantile unit root test, see Koenker and Xiao (2002, 2004). The complete tables with the computed statistics for carrying such test are in a working paper version of this article (Ferreira, 2007) or can be directly requested to the author.

Results reported in table 1 are coherent with previous works: unit root was only rejected for real exchange rate between the French franc and the Deutsch mark. The tests failed to reject the null of a unit root in the other series.

We also estimated $E(\alpha)^2$ by $\int_{\varepsilon}^{1-\varepsilon}\alpha(\tau)^2\,d\tau$ using $\tau \in [0.05, 0.95]$. The results are in the last row of table 1. Consistent with the previous two tests, the lowest value was obtained for RER$^{fr/gr}$ (0.934). The value for RER$^{it/uk}$ was very similar (0.938). Results closer to 1 were observed for RER$^{it/fr}$ and RER$^{it/gr}$: 0.991 and 0.981, respectively. Despite the absence of critical values to compare these estimates, the statistics suggest that RER$^{fr/gr}$ and RE$^{it/uk}$ behave as a stationary series in the long run. Very likely we would conclude that RER$^{it/fr}$ and RER$^{it/gr}$ feature a unit root given the proximity of $\int_{\varepsilon}^{1-\varepsilon}\alpha(\tau)^2\,d\tau$ to 1. It is harder to comment on RER$^{fr/uk}$ and RER$^{uk/gr}$ given their intermediate values.

Table 2 reports the estimated value of $\alpha_1(\tau)$ for deciles and whether unit root was rejected. The results show different patterns among the series analyzed. We verify, for instance, that a local random walk behavior is rejected in lower deciles of RER$^{it/fr}$, RER$^{it/gr}$, and RER$^{fr/gr}$, but a local explosive behavior - $\alpha_1(\tau) > 1$ - is found at higher deciles. The pattern is different for relations involving the British pound, in which case the estimated coefficients are closer to each other and do not become higher than 1.

These differences mean that scale does not seem an important issue for relations involving the pound, implying that when plotting quantile functions one should expect the fitted lines to be parallel to each other, since their slopes are similar.

On the other hand, location and scaling may be both relevant to model the other three relations, and, since the slope coefficients increase monotonically, one should expect observing a fan shaped scatter plot after plotting each estimated quantile function.

These patterns are indeed observed in the left hand side graphs of figures 2, 3 and 4, where the fitting of the linear quantile models, corresponding to the results reported in table 2, are plotted. The quantile functions involving the pound seem parallel, suggesting a more symmetric adjustment, regardless of past values of the real exchange rate.

In the case of $\text{RER}^{it/fr}$, $\text{RER}^{it/gr}$, and $\text{RER}^{fr/gr}$, steeper slopes are observed in upper conditional quantiles forming a fan shaped graph. This reflects the real exchange rate can assume wider range in period t if a higher value is observed in $t-1$, indicating a type of heterokedasticity and asymmetry not yet explored in previous works on real exchange rates.

Such a heterokedasticity can be dealt with by a quadratic autoregressive specification[11] that is similar to the previous QAR (equation 8), but with y_{t-1}^2 included as a covariate and β introduced in the parameter vector. These modifications lead to $x_t = (1, y_{t-1}, y_{t-1}^2, \Delta y_{t-1}, \ldots, \Delta y_{t-q})^T$ and $\alpha(\tau) = (\alpha_0(\tau), \alpha_1, \beta, \alpha_2, \ldots, \alpha_{p+1})$.

The right hand side graphs in figures 2, 3, and 4 show the fit of the quadratic specifications. In figure 5 we plot the point estimates of the quadratic terms and their 90% confidence interval for every relation analyzed.

The first thing to notice, as expected, is that heteroskedasticity does not seem to be a problem for relations involving the pound, since the quadratic term is not significant (figure 5). This can be visually assessed by observing that the fittings of the linear and quadratic models are very similar for these relations.

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[11] The heterokedasticity obtained in our work is similar to that of the Sydney temperature analyzed by Koenker (2005). He suggested a quadratic model to deal with this problem.

Contrary to this, figure 5 shows that the quadratic coefficients are significant in several quantile functions of $RER^{it/fr}$, $RER^{it/gr}$, and $RER^{fr/gr}$. Again, visual inspection of figures 2 and 3 suggests the quadratic model does a nice job in the fitting of these relations. This also indicates that the quadratic specification seems appropriate to deal with the heterokedasticity in these time series.

INSERT FIGURE 2

INSERT FIGURE 3

INSERT FIGURE 4

INSERT FIGURE 5

Given the quadratic specification was more appropriate for some percentiles, we moved a step forward and chose the best model for each estimated equation. Modeling percentiles individually allows the characterization of the entire distribution followed by each time series in a very flexible manner. Without the rigid structure imposed by well behaved distribution functions, our exercise provides tools to verify whether a real exchange rate depicts an asymmetric behavior by obtaining the conditional probabilities for movements in RER.

INSERT TABLE 3

INSERT TABLE 4

INSERT TABLE 5

Verifying Asymmetries

Tables 3 to 8, the most central to this work, summarize our findings about asymmetric adjustment in the real exchange rates. The conclusions were drawn based on i) measures of condition dispersion and also on ii) conditional probabilities for each RER to move up and down. The appendix brings figures of the conditioned distribution functions, which helps visualizing these asymmetries.

Dispersion was analyzed in terms of estimated conditional standard deviation, $\hat{\sigma}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$, and estimated conditional range, $\hat{R}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$. The other measure that allows capturing asymmetry is the probability for the real exchange rate $\hat{q}_t$ to be above and below *z* standard deviations from a conditioned value q_{t-1}:

$\Pr(\hat{q}_t > \mid Q_{y_{t-1}}(\tau) \pm z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)) \mid)$.

Dispersion

$\hat{\sigma}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$ and $\hat{R}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$ are larger for relations involving the British pound. The estimated standard deviations situate around 0.015, 0.016 and 0.017 for $\text{RER}^{it/uk}$ (table 6), $\text{RER}^{fr/uk}$ (table 7), and $\text{RER}^{gr/uk}$ (table 8), respectively, regardless to where we condition q_{t-1}. Conditioning q_{t-1} at central quartiles delivers the following estimates for $\text{RER}^{it/fr}$ (table 3), $\text{RER}^{it/gr}$ (table 4), and $\text{RER}^{fr/gr}$ (table 5): 0.009, 0.008, and 0.005, respectively.

INSERT TABLE 6

INSERT TABLE 7

INSERT TABLE 8

Higher variability in $\hat{\sigma}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$ is observed for RER$^{it/gr}$, RER$^{it/f}$, and RER$^{fr/gr}$. This is illustrated by the fact that $\hat{\sigma}^{it/fr}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$ equals 0.005 for $Q_{q_{t-1}}(\tau = \{0.05, 0.1\})$, but then jumps to 0.14 and 0.15 for $Q_{q_{t-1}}(\tau = \{0.90, 0.95\})$, respectively. Similarly, $\hat{\sigma}^{it/gr}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$ equals 0.006 at $\tau = \{0.05, 0.1\}$ and 0.15 and 0.17 for $\tau = \{0.90, 0.95\}$, respectively. Though more stable, the same pattern is observed for RER$^{fr/gr}$, due mainly to the increase in the standard deviation at higher conditioned percentiles.

Analysis for the *estimated conditioned range* support those based on standard deviation: relations involving the pound have higher $\hat{R}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$, but are more stable. For the remaining series the values of $\hat{R}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$ are smaller but less stable. We observe, for instance, that the estimated range of RER$^{it/fr}$ and RER$^{it/gr}$, at low values of q_{t-1}, is about three times smaller than those computed at high q_{t-1}.

Probability of Appreciation and Devaluation

In tables 3-8, the label $\Pr\_L(z)$ refers to the estimated probability of q_t being *z* standard deviations below the conditioned value a period before, i.e.

$\Pr\_L(z) = \Pr(\hat{q}_t < Q_{y_{t-1}}(\tau) - z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$. Similarly, $\Pr\_U(z)$ is the probability that q_t will assume values z standard deviations above the conditioned value:

$\Pr\_U(z) = \Pr(\hat{q}_t > Q_{y_{t-1}}(\tau) + z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$. We compute these probabilities for $z = \{1, 1.5, 2\}$.

Relations involving the pound (tables 6, 7 and 8) have higher probability of depreciation (appreciation) given a very appreciated (depreciated) q_{t-1}. More similar probability for moving in either direction was observed when q_{t-1} assumed central values. These patterns, expected in case of mean reversion, were also observed for other relations, with some local exception.

For RER$^{fr/gr}$, $z=1$ and q_{t-1} at $\tau=0.05$ (i.e., very appreciated franc) we verified almost the same probabilities for moving in either direction: 16.5% for falling more than 1 standard deviation and 15.4% for increasing.

For RER$^{it/fr}$, $z=\{1,1.5\}$ and q_{t-1} conditioned at $\tau=0.05$ (i.e., very appreciated lira) we still observed a much higher probability for appreciation (25.3% for $z=1$ and 11% for $z=1.5$) than depreciation (7.7% for $z=1$ and 6.6% for $z=1.5$).

The analysis when $z=2$ is particularly interesting because it informs the probabilities of assuming very extreme values. Except for RER$^{fr/gr}$, the general pattern already mentioned remains: higher probability of appreciation (depreciation) given an extremely depreciated (appreciated) currency. Exception was again verified for RER$^{fr/gr}$, since the probability for the franc to depreciate was around 5% regardless of the conditioned value at $t-1$. For this relation, higher probability of appreciation (1.1%) was only observed when conditioning q_{t-1} at $\tau=0.75$.

Another issue to notice is the low probability for the RER$^{uk/gr}$ to assume very extreme values, contrasting with the estimates for the other relations.

The main finding of the previous analysis is that there is indeed higher probability of moving back to central values of the real exchange rate, which corroborates basic economic intuition that backs the PPP theory. However, we also notice asymmetry, since these probabilities are not necessarily the same even when the conditioned values are symmetrical relative to the median of the unconditional distribution of each time series.

Back to the Literature

The main message of the current work points in the direction that real exchange rate indeed tends to converge to central values. However, except for the case of the relation between the German mark and the French franc, our results do not allow concluding that there tends to be a reversal towards a specific mean.

Transaction cost theory says that the RER more likely depicts a random walk behavior at central values, while a convergent autoregressive pattern would be observed when it assumes extreme values. Results reported in tables 3-8 showing higher probability for appreciation (depreciation) if RER is depreciated (appreciated) favor the intuition behind the transaction cost theory, supporting the idea that convergence to central values, or to a central band, happens.

Obtaining several coefficients larger than or equal to one indicate that, at least locally, the real exchange rates may indeed depict a non-reverting property. These results may be behind the conclusions reached by Kim and Lima (2004), as they did not reject the hypothesis of a root near unity for RER of the G7 countries. The local random walk or explosive behavior may be the driven force behind the long memory of real exchange rates. The fact that local explosive behavior occurs even for RER franc/mark, a relation for which most standard tests reject the unit root, make this point more relevant.

From a pure statistical point of view, an important point raised by this paper has to do with the need to take heterokedasticity more seriously when dealing with real exchange rate time series. This problem was well observed in the work of Bec, Carrasco and Salem (2004) who rejected the null of homoskedasticity in the following real exchange rates: $RER^{gr/uk}$, $RER^{it/uk}$, $RER^{fr/gr}$, $RER^{it/gr}$, $RER^{it/fr}$. They proposed a method to correct for the presence of heterokedasticity, which allowed them to construct a unit root test to verify if a TAR model would best fit the data. Important to note, however, that stability in the conditioned standard deviations for relations involving the British pound suggests homoskedasticity in these relations, contradicting, in this sense, that $RER^{gr/uk}$, $RER^{it/uk}$ are heterokedastic.

Another important issue, raised by the work of Leon and Najarian (2005) and corroborated by ours, has to do with the use of symmetric TAR models. Since we verified that the probability of appreciating conditioned on depreciated RER is not necessarily the same as the probability of depreciating given an appreciated currency, it seems dangerous relying on symmetric TAR for modeling the stochastic process followed by real exchange rate.

V. Conclusions

Quantile autoregression was used to analyze the behavior of pair wise real exchange rate between the Italian lire, French franc, Deutsch mark, and the British pound, using data from January of 1973 to December of 1998.

The main contribution of this paper is the identification and measurement of two sources of asymmetry in the adjustment process of RER. The first is the heterokedasticity: dispersion, measured in terms of standard deviation and range, varies with the conditioned value of the real exchange rate. The second refers to the probabilities of changes in real exchange rates: the probability of increases and falls of a RER changes according to the extreme conditioned value, i.e., there is higher probability for the real exchange rate to appreciate (depreciate) given the currency is depreciated (appreciated). Also important is the found that these probabilities were not symmetric.

These findings support the idea that exchange rates tend towards central intervals when they are very far from them, suggesting that transaction cost theory may be a good story for explaining the patterns also captured by the use of TAR. However, our results do not necessarily validate the use of symmetric thresholds so commonly employed in the TAR literature when modeling real exchange rate. The analysis carried gives a strong support to non symmetric TAR models as implemented by Leon and Najarian (2005).

Another novelty was the use of a quadratic autoregressive specification, at some quantile functions, to deal with heterokedasticity in $RER^{it/gr}$, $RER^{it/fr}$, and $RER^{fr/gr}$. The linear specification was on general superior for relations against the pound, for which we also observed higher dispersion (measured in terms of standard deviation and range).

Although we have not explored any issue related to optimal currency area, the fact that conditioned standard deviations are smaller for relations between the lire, franc, and mark leaves the impression that different adjustment of the pound may justify why it did not join in currency area with the other three currencies. Since this is simply a primary thought, it would be interesting analyzing this issue in light of the new information provided by our exercise.

Following the same stream, it would still be interesting to estimate models with economic meaningful covariates as a way to better understand the reasons behind the asymmetries exposed by this work.

Appendix: Conditional Predicted Density Function

Above each density plot we have the value of q_{t-1} at selected quantiles. At the bottom we report the probability of $\hat{q}_t$ being smaller and greater than q_{t-1}. The vertical line in each plot corresponds to the value of q_{t-1}.

INSERT FIGURE 6

INSERT FIGURE 7

INSERT FIGURE 8

INSERT FIGURE 9

INSERT FIGURE 10

Acknowledgement. I am indebted to Roger Koenker for his advises and comments in previous versions of this work. I am also indebted to Antonio Galvão for sharing codes and knowledge. As usual, all errors are my own responsibility.

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FIGURES

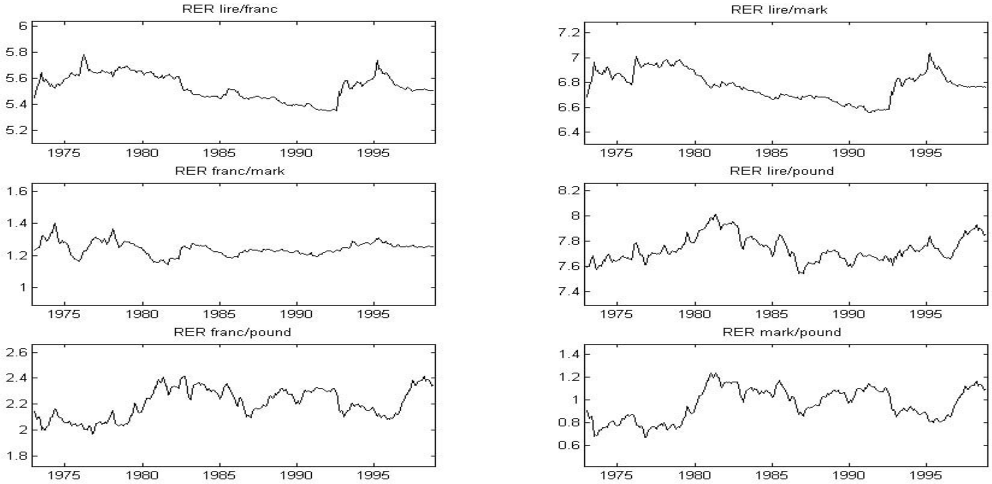

Figure 1. Logarithm of the real exchange rate between Italian lire, French franc, Deutsch mark, and the British pound from Jan/1973 to Dec/1998.

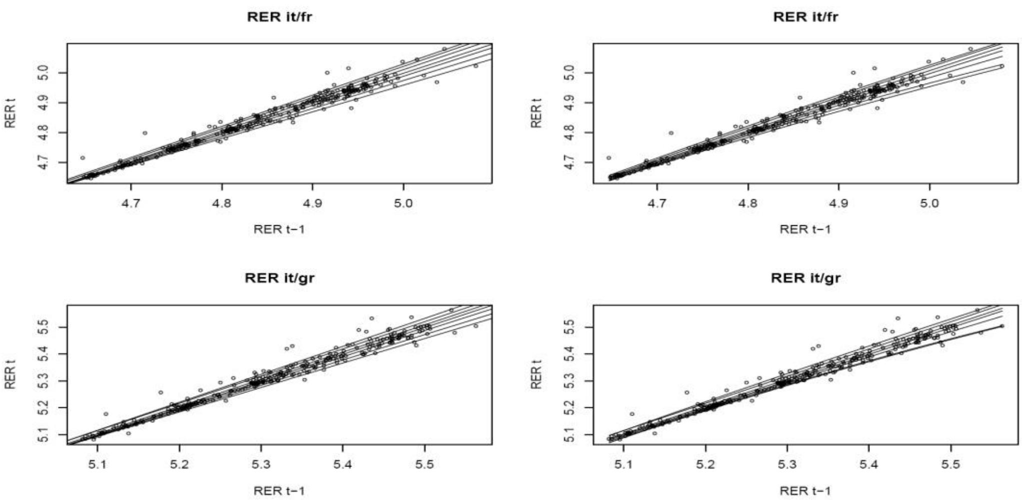

Figure 2: Fitted lines for linear (left) and quadratic (right) models of some quantile autoregression: $RER^{it/fr}$ and $RER^{it/gr}$.

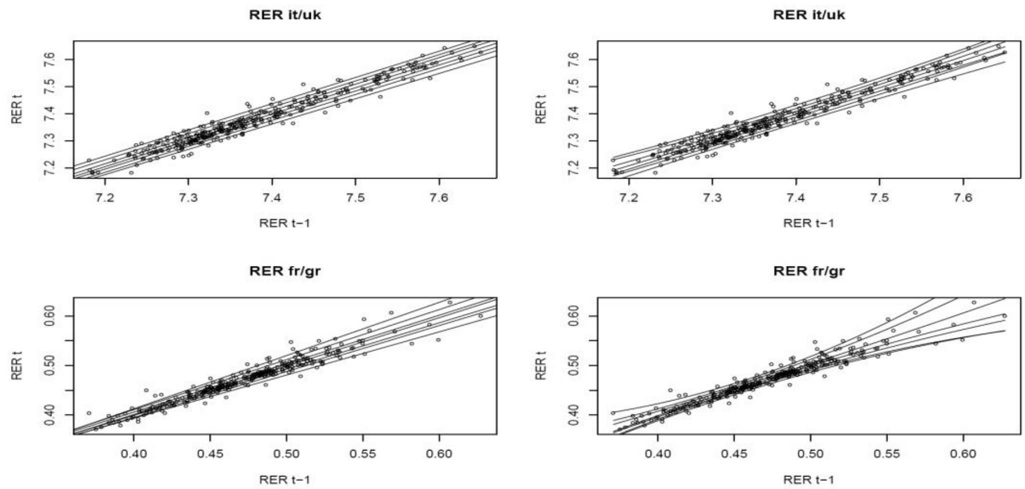

Figure 3. Fitted lines for linear (left) and quadratic (right) models of some quantile autoregression: $RER^{it/uk}$ and $RER^{fr/gr}$.

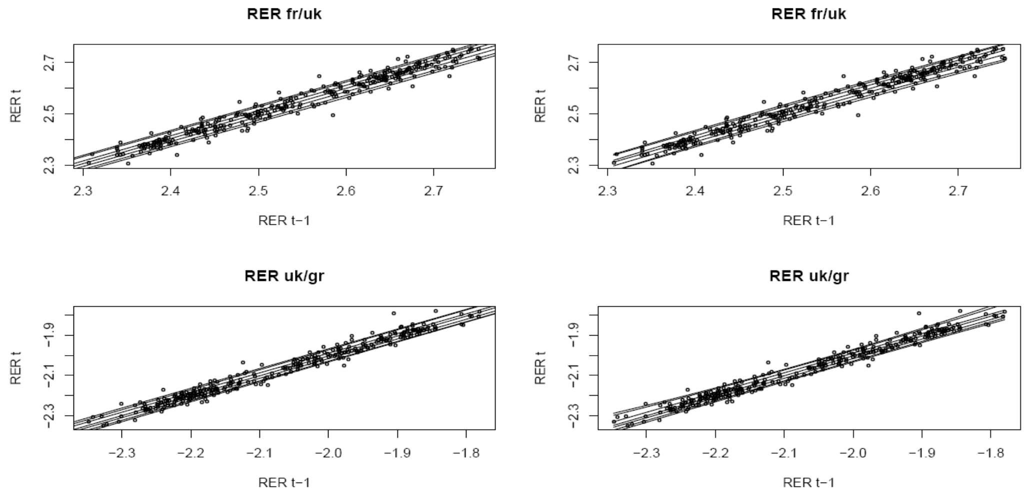

Figure 4. Fitted lines for linear (left) and quadratic (right) models of some quantile autoregression: $RER^{fr/uk}$ and $RER^{uk/gr}$.

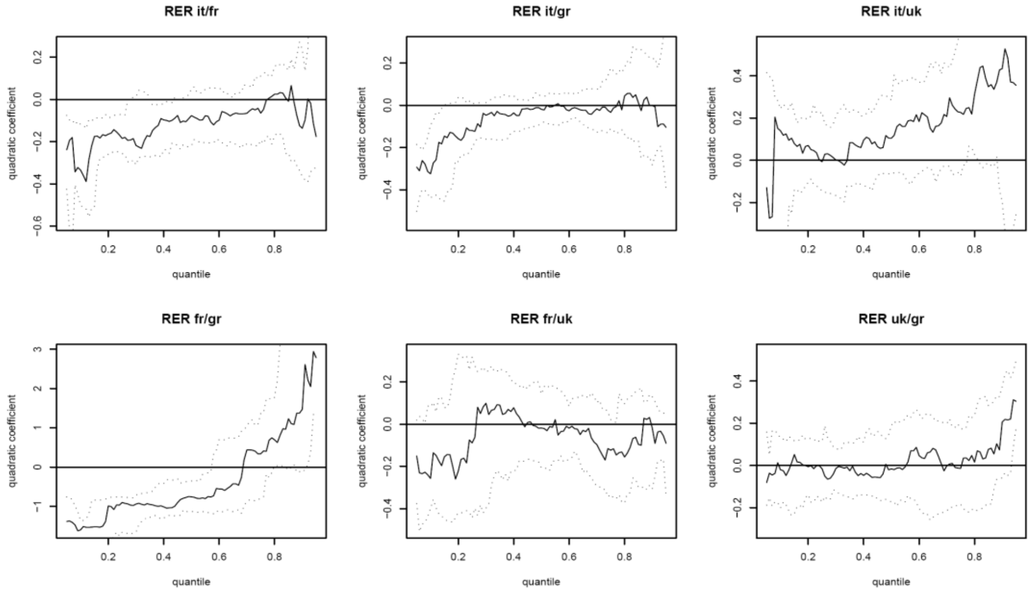

Figure 5. Estimated quadratic coefficients and their respective 90% confidence interval.

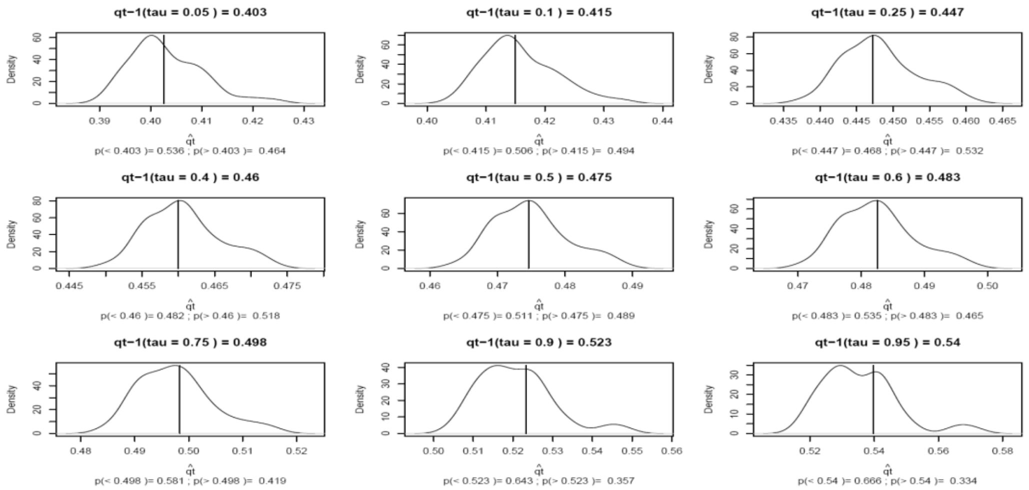

Figure 6. Predicted density of $RER^{fr/gr}$ using the best specification at each τ .

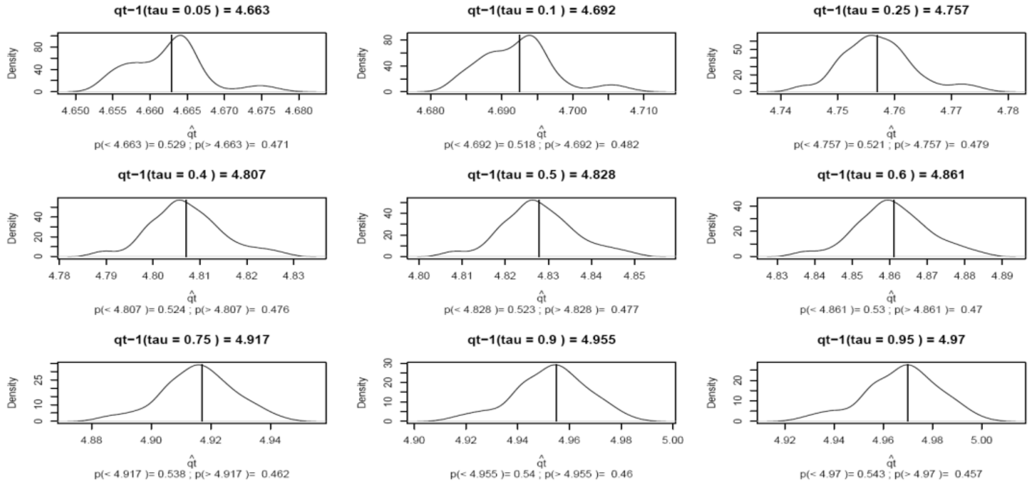

Figure 7. Predicted density of $RER^{it/fr}$ using the best specification at each τ . (dens_final_if.pdf)

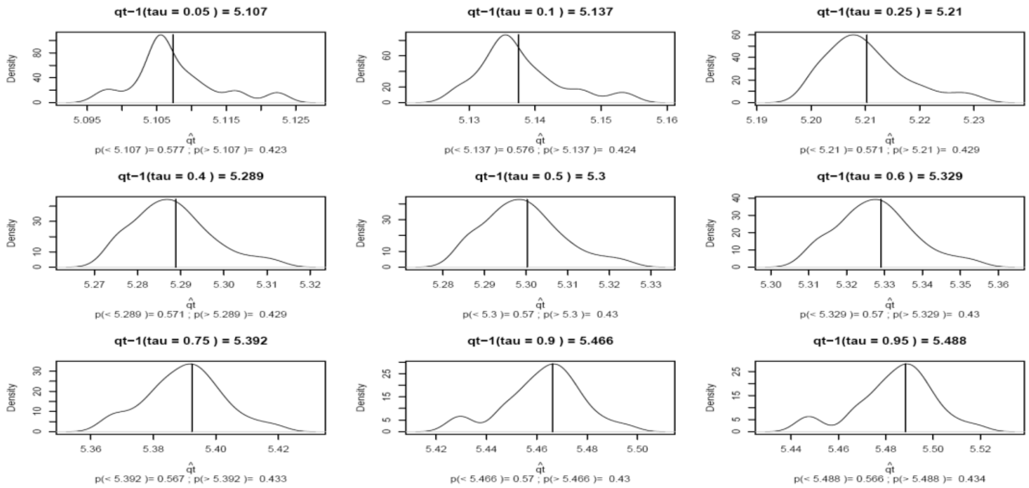

Figure 8. Predicted density of $RER^{it/gr}$ using the best specification at each τ .

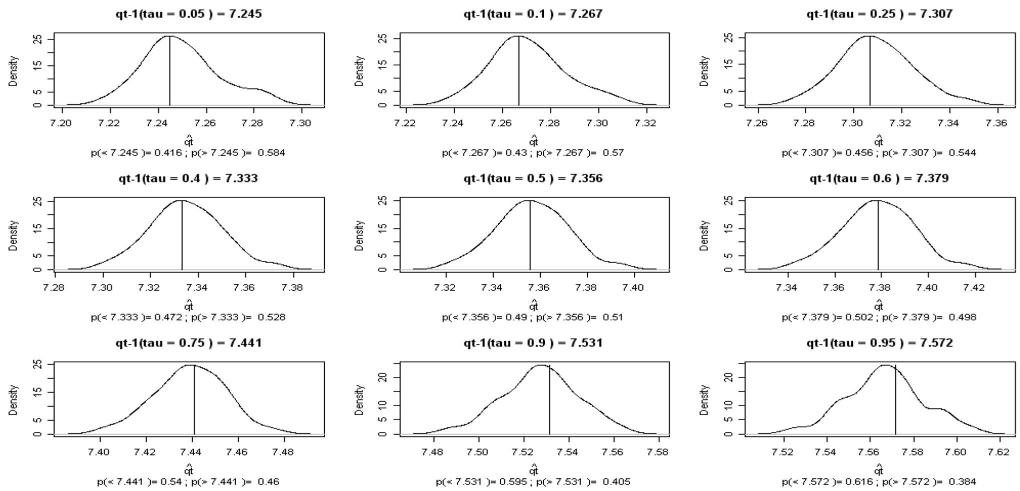

Figure 9. Predicted density of $RER^{it/uk}$ using the best specification at each τ . (dens_final_iuk.pdf)

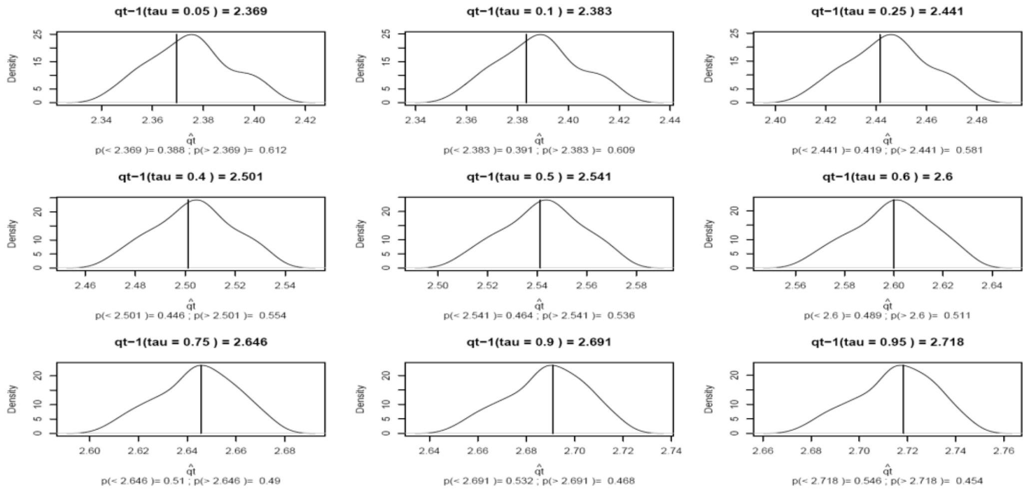

Figure 10. Predicted density of $RER^{fr/uk}$ using the best specification at each τ .

TABLES

Table 1: Test statistics of the ADF and PP unit root tests and estimated value of $E(\alpha)^2$.

Test	$RER^{it/fr}$	$RER^{it/gr}$	$RER^{fr/gr}$	$RER^{it/uk}$	$RER^{fr/uk}$	$RER^{uk/gr}$
ADF	-1.95	-1.98	-3.28*	-2.76	-2.35	-2.07
PP	-1.9	-1.86	-2.90*	-2.45	-1.98	-1.87
$E(\alpha)^2$	0.991	0.981	0.934	0.938	0.953	0.965

* stands for rejection at 5% level of significance.

Table 2: Estimated values of $\alpha_1(\tau)$ and quantile unit root test.

Deciles	$RER^{it/fr}$	$RER^{it/gr}$	$RER^{fr/gr}$	$RER^{it/uk}$	$RER^{fr/uk}$	$RER^{uk/gr}$
10^{th}	0.938*	0.942*	0.929	0.959	0.962	0.991
20^{th}	0.967*	0.966*	0.933*	0.956	0.981	0.986
30^{th}	0.983	0.976*	0.947*	0.976	0.981	0.980
40^{th}	0.992	0.987	0.965*	0.973	0.978	0.980
50^{th}	0.994	0.998	0.978	0.973	0.978	0.987
60^{th}	1.002	1.002	0.992	0.978	0.985	0.980
70^{th}	1.016	1.007	0.976	0.974	0.983	0.978
80^{th}	1.032	1.012	0.984	0.969	0.967	0.979
90^{th}	1.035	1.011	1.012	0.971	0.969	0.978

* stands for rejection at 5% significance level. See appendix for details about quantile unit root test.

Table 3: Quantifying the asymmetry in the $RER^{it/fr}$

$Q_{q_{t-1}}(\tau)$	$\hat{\sigma}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$	$\hat{R}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$	$\Pr\_L(z=1)$	$\Pr\_U(z=1)$	$\Pr\_L(z=1.5)$	$\Pr\_U(z=1.5)$	$\Pr\_L(z=2)$	$\Pr\_U(z=2)$
$4.66_{\tau=0.05}$	0.005	0.023	0.253	0.077	0.110	0.066	0.000	0.055
$4.69_{\tau=0.1}$	0.005	0.025	0.165	0.110	0.066	0.066	0.000	0.055
$4.76_{\tau=0.25}$	0.006	0.032	0.143	0.132	0.033	0.077	0.022	0.055
$4.81_{\tau=0.4}$	0.008	0.039	0.132	0.154	0.044	0.077	0.033	0.044
$4.83_{\tau=0.5}$	0.008	0.042	0.154	0.154	0.055	0.077	0.033	0.044
$4.86_{\tau=0.6}$	0.010	0.047	0.143	0.154	0.066	0.077	0.033	0.022
$4.92_{\tau=0.75}$	0.012	0.057	0.154	0.132	0.088	0.055	0.044	0.022
$4.95_{\tau=0.9}$	0.014	0.065	0.165	0.110	0.088	0.033	0.044	0.000
$4.97_{\tau=0.95}$	0.015	0.069	0.165	0.110	0.099	0.022	0.055	0.000

$\Pr\_L(z) = \Pr(\hat{q}_t < Q_{y_{t-1}}(\tau) - z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$, $\Pr\_U(z) = \Pr(\hat{q}_t > Q_{y_{t-1}}(\tau) + z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$

Table 4: Quantifying the asymmetry in the $RER^{it/gr}$

$Q_{q_{t-1}}(\tau)$	$\hat{\sigma}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$	$\hat{R}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$	$\Pr\_L(z=1)$	$\Pr\_U(z=1)$	$\Pr\_L(z=1.5)$	$\Pr\_U(z=1.5)$	$\Pr\_L(z=2)$	$\Pr\_U(z=2)$
$5.11_{\tau=0.05}$	0.006	0.027	0.121	0.143	0.044	0.099	0.000	0.066
$5.14_{\tau=0.10}$	0.006	0.027	0.099	0.154	0.022	0.099	0.000	0.066
$5.21_{\tau=0.25}$	0.007	0.031	0.165	0.143	0.011	0.088	0.000	0.066
$5.29_{\tau=0.40}$	0.009	0.039	0.165	0.143	0.055	0.077	0.000	0.044
$5.30_{\tau=0.50}$	0.009	0.040	0.165	0.143	0.066	0.077	0.000	0.033
$5.33_{\tau=0.60}$	0.010	0.044	0.165	0.132	0.099	0.077	0.000	0.033
$5.39_{\tau=0.75}$	0.012	0.055	0.187	0.099	0.099	0.055	0.044	0.022
$5.47_{\tau=0.90}$	0.015	0.070	0.198	0.088	0.088	0.033	0.088	0.000
$5.49_{\tau=0.95}$	0.017	0.075	0.198	0.077	0.088	0.033	0.088	0.000

$\Pr\_L(z) = \Pr(\hat{q}_t < Q_{y_{t-1}}(\tau) - z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$, $\Pr\_U(z) = \Pr(\hat{q}_t > Q_{y_{t-1}}(\tau) + z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$

Table 5: Quantifying the asymmetry in the $RER^{fr/gr}$

$Q_{q_{t-1}}(\tau)$	$\hat{\sigma}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$	$\hat{R}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$	Pr_$L(z=1)$	Pr_$U(z=1)$	Pr_$L(z=1.5)$	Pr_$U(z=1.5)$	Pr_$L(z=2)$	Pr_$U(z=2)$
$0.40_{\tau=0.05}$	0.007	0.033	0.165	0.154	0.022	0.055	0.000	0.055
$0.41_{\tau=0.10}$	0.006	0.029	0.143	0.187	0.022	0.110	0.000	0.055
$0.45_{\tau=0.25}$	0.005	0.023	0.099	0.198	0.022	0.132	0.000	0.077
$0.46_{=0.40}$	0.005	0.023	0.110	0.187	0.022	0.132	0.000	0.066
$0.47_{\tau=0.50}$	0.005	0.025	0.154	0.165	0.033	0.110	0.000	0.066
$0.48_{\tau=0.60}$	0.006	0.027	0.187	0.154	0.033	0.088	0.000	0.044
$0.50_{\tau=0.75}$	0.007	0.031	0.209	0.132	0.044	0.088	0.011	0.044
$0.52_{\tau=0.90}$	0.009	0.044	0.253	0.088	0.077	0.055	0.000	0.055
$0.54_{\tau=0.95}$	0.012	0.054	0.275	0.077	0.110	0.055	0.000	0.055

Pr_ $L(z) = \Pr(\hat{q}_t < Q_{y_{t-1}}(\tau) - z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$, Pr_ $U(z) = \Pr(\hat{q}_t > Q_{y_{t-1}}(\tau) + z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$

Table 6: Quantifying the asymmetry in the $RER^{it/uk}$

$Q_{q_{t-1}}(\tau)$	$\hat{\sigma}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$	$\hat{R}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$	Pr_$L(z=1)$	Pr_$U(z=1)$	Pr_$L(z=1.5)$	Pr_$U(z=1.5)$	Pr_$L(z=2)$	Pr_$U(z=2)$
$7.24_{\tau=0.05}$	0.016	0.072	0.088	0.198	0.022	0.154	0.000	0.088
$7.27_{\tau=0.10}$	0.016	0.072	0.099	0.198	0.033	0.143	0.000	0.077
$7.31_{\tau=0.25}$	0.015	0.072	0.110	0.198	0.044	0.088	0.011	0.033
$7.33_{\tau=0.40}$	0.015	0.073	0.132	0.176	0.055	0.077	0.022	0.033
$7.36_{\tau=0.50}$	0.015	0.074	0.154	0.154	0.066	0.055	0.022	0.033
$7.38_{\tau=0.60}$	0.015	0.074	0.165	0.132	0.077	0.044	0.033	0.033
$7.44_{\tau=0.75}$	0.015	0.077	0.187	0.121	0.088	0.044	0.033	0.011
$7.53_{\tau=0.90}$	0.017	0.080	0.231	0.121	0.121	0.033	0.033	0.000
$7.57_{\tau=0.95}$	0.018	0.084	0.231	0.132	0.121	0.033	0.033	0.000

Pr_ $L(z) = \Pr(\hat{q}_t < Q_{y_{t-1}}(\tau) - z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$, Pr_ $U(z) = \Pr(\hat{q}_t > Q_{y_{t-1}}(\tau) + z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$

Table 7: Quantifying the asymmetry in the $RER^{fr/uk}$

$Q_{q_{t-1}}(\tau)$	$\hat{\sigma}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$	$\hat{R}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$	Pr_$L(z=1)$	Pr_$U(z=1)$	Pr_$L(z=1.5)$	Pr_$U(z=1.5)$	Pr_$L(z=2)$	Pr_$U(z=2)$
$2.37_{\tau=0.05}$	0.016	0.063	0.099	0.231	0.033	0.176	0.000	0.066
$2.38_{\tau=0.10}$	0.016	0.063	0.099	0.220	0.044	0.176	0.000	0.055
$2.44_{\tau=0.25}$	0.016	0.064	0.132	0.209	0.044	0.121	0.000	0.044
$2.50_{\tau=0.40}$	0.016	0.065	0.165	0.198	0.055	0.110	0.000	0.011
$2.54_{\tau=0.50}$	0.016	0.065	0.176	0.187	0.066	0.077	0.022	0.011
$2.60_{\tau=0.60}$	0.016	0.066	0.187	0.176	0.099	0.066	0.022	0.000
$2.65_{\tau=0.75}$	0.016	0.067	0.209	0.143	0.110	0.055	0.022	0.000
$2.69_{\tau=0.90}$	0.016	0.067	0.220	0.121	0.121	0.044	0.044	0.000
$2.72_{\tau=0.95}$	0.016	0.068	0.231	0.110	0.121	0.044	0.055	0.000

Pr_ $L(z) = \Pr(\hat{q}_t < Q_{y_{t-1}}(\tau) - z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$, Pr_ $U(z) = \Pr(\hat{q}_t > Q_{y_{t-1}}(\tau) + z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$

Table 8: Quantifying the asymmetry in the $RER^{uk/gr}$

$Q_{q_{t-1}}(\tau)$	$\hat{\sigma}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$	$\hat{R}(\hat{q}_t \mid Q_{q_{t-1}}(\tau))$	Pr_$L(z=1)$	Pr_$U(z=1)$	Pr_$L(z=1.5)$	Pr_$U(z=1.5)$	Pr_$L(z=2)$	Pr_$U(z=2)$
$-2.26_{\tau=0.05}$	0.017	0.075	0.165	0.198	0.044	0.099	0.000	0.022
$-2.24_{\tau=0.10}$	0.017	0.072	0.165	0.198	0.044	0.099	0.000	0.022
$-2.20_{\tau=0.25}$	0.017	0.067	0.165	0.187	0.066	0.099	0.000	0.022
$-2.14_{\tau=0.40}$	0.016	0.063	0.176	0.176	0.077	0.099	0.000	0.000
$-2.08_{\tau=0.50}$	0.016	0.062	0.198	0.165	0.099	0.088	0.000	0.000
$-2.02_{\tau=0.60}$	0.016	0.062	0.231	0.154	0.110	0.077	0.011	0.000
$-1.95_{\tau=0.75}$	0.017	0.063	0.242	0.143	0.121	0.077	0.011	0.000
$-1.88_{\tau=0.90}$	0.017	0.069	0.253	0.132	0.121	0.044	0.011	0.000
$-1.86_{\tau=0.95}$	0.017	0.070	0.253	0.132	0.132	0.044	0.011	0.022

Pr_ $L(z) = \Pr(\hat{q}_t < Q_{y_{t-1}}(\tau) - z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$, Pr_ $U(z) = \Pr(\hat{q}_t > Q_{y_{t-1}}(\tau) + z\sigma(\hat{q}_t \mid Q_{y_{t-1}}(\tau)))$