CAPÍTULO 9 - AULA 2

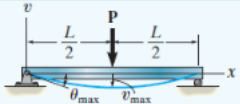
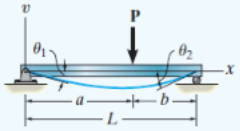
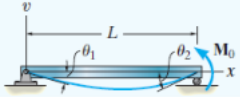
DEFLEXÃO DE VIGAS E EIXOS - MÉTODO DA SUPERPOSIÇÃO

17 de novembro de 2026

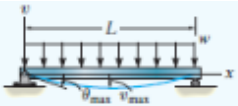
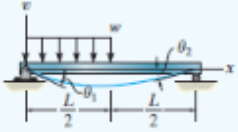
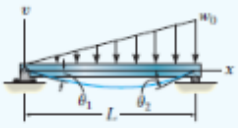
Objetivo

- Determinar a inclinação e a deflexão em pontos de vigas submetidas a múltiplos carregamentos, utilizando tabelas de casos fundamentais e o princípio da superposição dos efeitos.

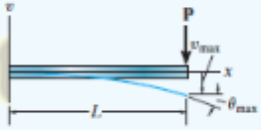
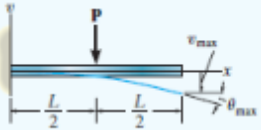
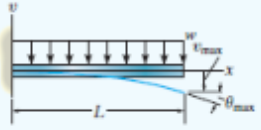
Tabelas - Vigas biapoiadas

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{16EI}$	$v_{max} = \frac{-PL^3}{48EI}$	$v = \frac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$
	$\theta_1 = \frac{-Pab(L+b)}{6EIL}$ $\theta_2 = \frac{Pab(L+a)}{6EIL}$	$v\Big\vert_{x=a} = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\theta_1 = \frac{-M_0 L}{6EI}$ $\theta_2 = \frac{M_0 L}{3EI}$	$v_{max} = \frac{-M_0 L^2}{9\sqrt{3}\,EI}$ at $x = 0.5774L$	$v = \frac{-M_0 x}{6EIL}(L^2 - x^2)$

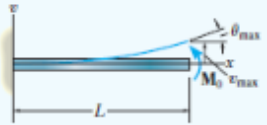
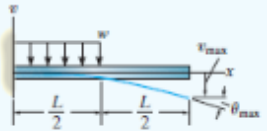
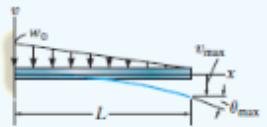
Tabelas - Vigas biapoiadas

	$\theta_{max} = \frac{-wL^3}{24EI}$	$v_{max} = \frac{-5wL^4}{384EI}$	$v = \frac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \frac{-3wL^3}{128EI}$ $\theta_2 = \frac{7wL^3}{384EI}$	$v\Big\|_{x=L/2} = \frac{-5wL^4}{768EI}$ $v_{max} = -0.006563\frac{wL^4}{EI}$ at $x = 0.4598L$	$v = \frac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \frac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \frac{-7w_0L^3}{360EI}$ $\theta_2 = \frac{w_0L^3}{45EI}$	$v_{max} = -0.00652\frac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \frac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

Tabelas - Vigas engastadas-livres

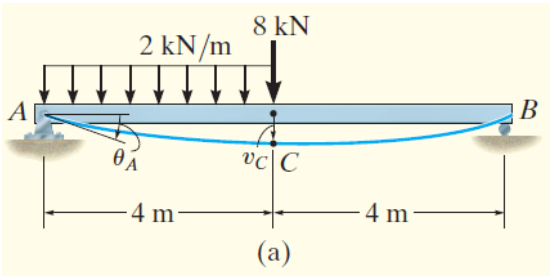
Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{2EI}$	$v_{max} = \frac{-PL^3}{3EI}$	$v = \frac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \frac{-PL^2}{8EI}$	$v_{max} = \frac{-5PL^3}{48EI}$	$v = \frac{-Px^2}{12EI}(3L - 2x) \quad 0 \le x \le L/2$ $v = \frac{-PL^2}{48EI}(6x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \frac{-wL^3}{6EI}$	$v_{max} = \frac{-wL^4}{8EI}$	$v = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$

Tabelas - Vigas engastadas-livres

	$\theta_{max} = \frac{M_0 L}{EI}$	$v_{max} = \frac{M_0 L^2}{2EI}$	$v = \frac{M_0 x^2}{2EI}$
	$\theta_{max} = \frac{-wL^3}{48EI}$	$v_{max} = \frac{-7wL^4}{384EI}$	$v = \frac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right)$ $0 \le x \le L/2$ $v = \frac{-wL^3}{384EI}(8x - L)$ $L/2 \le x \le L$
	$\theta_{max} = \frac{-w_0 L^3}{24EI}$	$v_{max} = \frac{-w_0 L^4}{30EI}$	$v = \frac{-w_0 x^2}{120EIL}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$

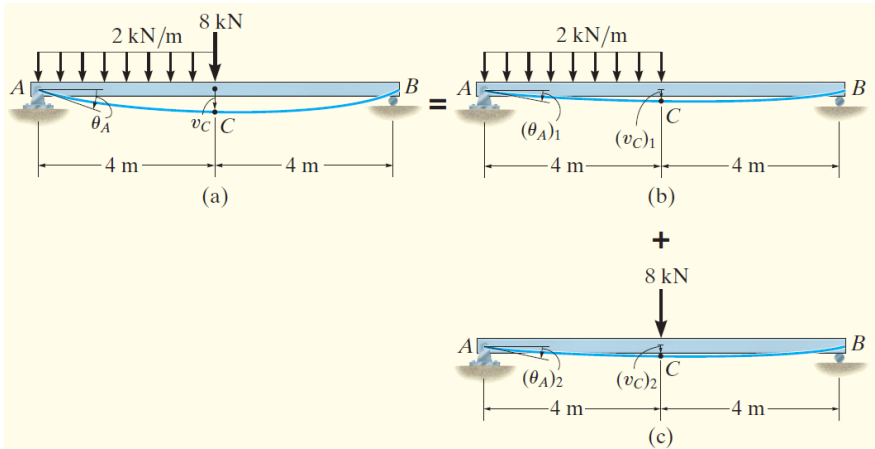
Exemplo 1

Determine o deslocamento no ponto C e a inclinação no apoio A da viga apresentada na Figura. Considere *EI* constante.

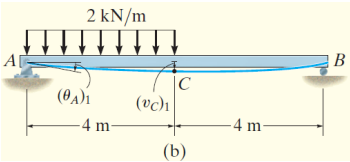


Exemplo 1

Determine o deslocamento no ponto C e a inclinação no apoio A da viga apresentada na Figura. Considere *EI* constante.

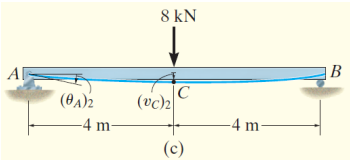


Exemplo 1



$$(\theta_A)_1 = \frac{3wL^3}{128EI} = \frac{3(2\,\text{kN/m})(8\,\text{m})^3}{128EI} = \boxed{\frac{24\,\text{kN}\cdot\text{m}^2}{EI}}$$

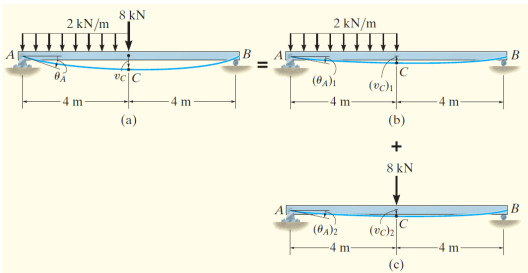
$$(v_C)_1 = \frac{5wL^4}{768EI} = \frac{5(2\,\text{kN/m})(8\,\text{m})^4}{768EI} = \boxed{\frac{53{,}33\,\text{kN}\cdot\text{m}^3}{EI}}$$



$$(\theta_A)_2 = \frac{PL^2}{16EI} = \frac{8\,\text{kN}(8\,\text{m})^2}{16EI} = \boxed{\frac{32\,\text{kN}\cdot\text{m}^2}{EI}}$$

$$(v_C)_2 = \frac{PL^3}{48EI} = \frac{8\,\text{kN}(8\,\text{m})^3}{48EI} = \boxed{\frac{85{,}33\,\text{kN}\cdot\text{m}^3}{EI}}$$

Exemplo 1



$$(+\circlearrowleft) \quad \theta_A = (\theta_A)_1 + (\theta_A)_2$$

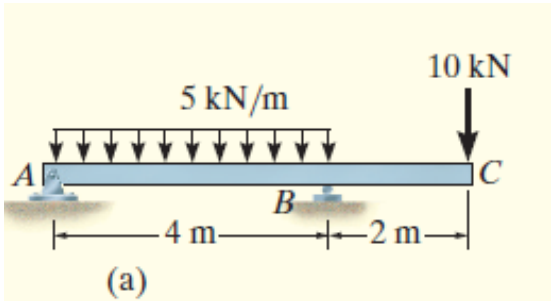
$$\boxed{\theta_A = \frac{56\,\text{kN}\cdot\text{m}^2}{EI}} \qquad \text{Ans.}$$

$$(+\downarrow) \quad v_C = (v_C)_1 + (v_C)_2$$

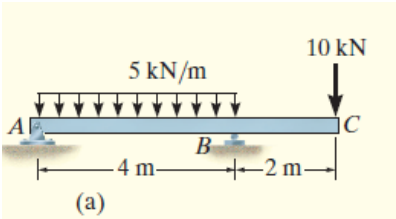
$$\boxed{v_C = \frac{139\,\text{kN}\cdot\text{m}^3}{EI}} \qquad \text{Ans.}$$

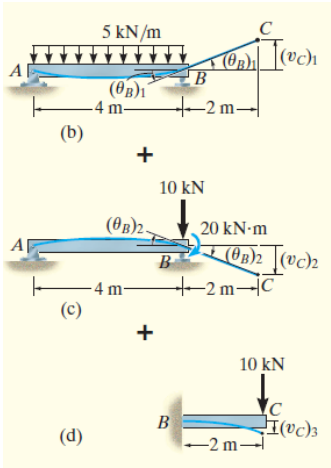
Exemplo 2

Determine o deslocamento vertical do ponto C. Considere *EI* constante.

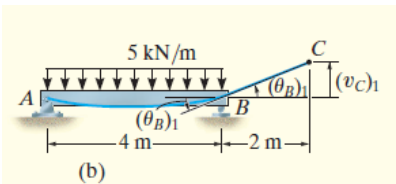


Exemplo 2


10 kN
5 kN/m
A
B
C
4 m
2 m
(a)


5 kN/m
C
A
B
$(\theta_B)_1$
$(v_C)_1$
$(\theta_B)_1$
4 m
2 m
(b)
+
10 kN
$(\theta_B)_2$
20 kN·m
A
B
$(\theta_B)_2$
$(v_C)_2$
4 m
2 m
C
(c)
+
10 kN
B
C
$(v_C)_3$
2 m
(d)

Exemplo 2



1. Cálculo da inclinação no ponto B

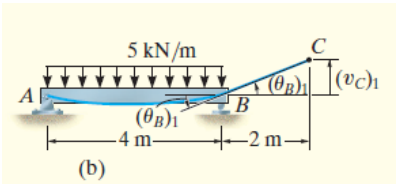
Inicialmente, calcula-se a inclinação no ponto B, provocada pela carga distribuída atuante no trecho biapoiado da viga.

$$(\theta_B)_1 = \frac{wL^3}{24EI}$$

Substituindo os valores:

$$(\theta_B)_1 = \frac{5\ \text{kN/m}\,(4\ \text{m})^3}{24EI} = \frac{13{,}33\ \text{kN}\cdot\text{m}^2}{EI}$$

Exemplo 2



2. Cálculo do deslocamento vertical no ponto C

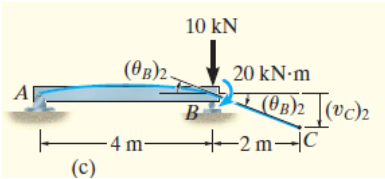
Como a inclinação é pequena, o deslocamento vertical do ponto C pode ser determinado pela aproximação geométrica:

$$(v_C)_1 = (2\,\text{m})(\theta_B)_1$$

Substituindo:

$$(v_C)_1 = 2\,\text{m}\left(\frac{13{,}33\ \ \text{kN}\cdot\text{m}^2}{EI}\right) = \frac{26{,}67\ \ \text{kN}\cdot\text{m}^3}{EI}$$

Exemplo 2



1. Efeito da carga concentrada no apoio B

A carga de 10 kN aplicada no balanço pode ser substituída, no apoio B do trecho biapoiado, por um sistema estaticamente equivalente, constituído por uma força de 10 kN e um momento de binário de 20 kN · m.

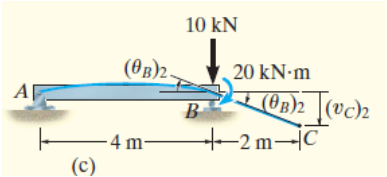
A força de 10 kN não provoca inclinação em B. Entretanto, o momento de binário de 20 kN · m provoca uma inclinação, dada por:

$$(\theta_B)_2 = \frac{M_0 L}{3EI}$$

Substituindo os valores:

$$(\theta_B)_2 = \frac{20\,\text{kN}\cdot\text{m}\,(4\,\text{m})}{3EI} = \frac{26{,}67\,\text{kN}\cdot\text{m}^2}{EI}$$

Exemplo 2



2. Cálculo do deslocamento vertical no ponto C

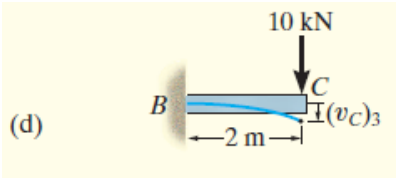
O deslocamento vertical em C é determinado pela rotação da seção em B, considerando pequenos ângulos:

$$(v_C)_2 = (2\,\text{m})(\theta_B)_2$$

Substituindo:

$$(v_C)_2 = 2\,\text{m}\left(\frac{26{,}67\,\text{kN}\cdot\text{m}^2}{EI}\right) = \frac{53{,}33\,\text{kN}\cdot\text{m}^3}{EI}$$

Exemplo 2



1. Efeito da carga concentrada no trecho BC

O trecho em balanço BC sofre deslocamento vertical devido à ação da força concentrada de 10 kN aplicada em sua extremidade livre, conforme a Figura.

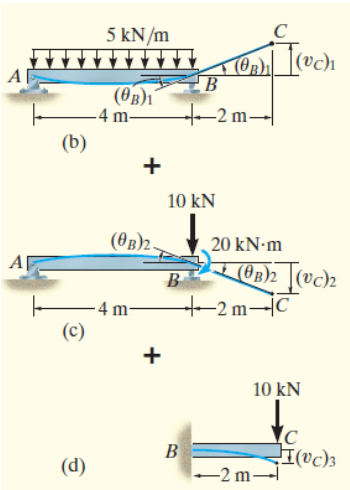
O deslocamento vertical em C é dado por:

$$(v_C)_3 = \frac{PL^3}{3EI}$$

Substituindo os valores:

$$(v_C)_3 = \frac{10\,\text{kN}\,(2\,\text{m})^3}{3EI} = \frac{26{,}67\,\text{kN}\cdot\text{m}^3}{EI}$$

Exemplo 2



Determinação do deslocamento vertical resultante em C

Somando algebricamente as contribuições dos três efeitos analisados anteriormente, obtém-se:

$$v_C = (v_C)_1 + (v_C)_2 + (v_C)_3$$

$$v_C = -\frac{26{,}7}{EI} + \frac{53{,}3}{EI} + \frac{26{,}7}{EI}$$

$$v_C = \frac{53{,}3\,\text{kN}\cdot\text{m}^3}{EI} \quad \uparrow$$